Extensive Form Games with Perfect Information

Levent Koçkesen

1 Extensive Form Games

So far we have assumed that players, when taking their actions, either did so simultaneously, or without knowing the action choice of the other players. Although, this modelling assumption might be appropriate in some settings, there are many situations in the world of business and politics that involve players moving sequentially after observing what the other players have done. For example, a bargaining situation between a seller and a buyer may involve the buyer making an offer and the seller, after observing the buyer's offer, either accepting or rejecting it. Or imagine an incumbent senator deciding whether to run an expensive ad campaign for the upcoming elections and a potential challenger deciding whether to enter the race or not, after observing the campaign decision of the incumbent. Both of these situations involve a player choosing an action after observing the action of the other player.

The **extensive form** of a game, as opposed to the strategic form, provides a more appropriate framework to analyze certain interesting questions that arise in strategic interactions that involve sequential moves.

1.1 Game Trees

As you now very well know, strategic form of a game has three ingredients: (1) the set of players, (2) the set of actions, and (3) the payoff functions. The extensive form provides a richer specification of a strategic interaction by specifying **who** moves **when** doing **what** and with **what information**. The easiest way to represent an extensive form game is to use a **game tree**, which is a multi-person generalization of a decision tree.

To illustrate, let us go back to the bargaining example above and assume that the buyer moves first by offering either \$500 or \$100 for a product that she values \$600. The seller, for whom the value of the object is \$50, responds by either accepting (A) or rejecting (R) the offer. We can represent this situation by the game tree in Figure 1.

Game trees are made up of

- nodes
- branches
- information sets

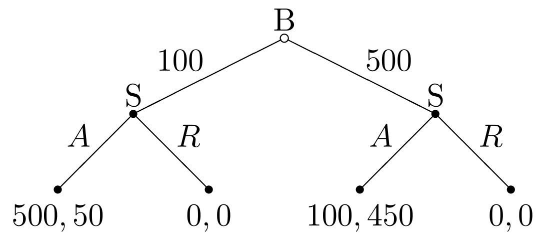

Figure 1: Bargaining Game

- player labels
- action labels

and

- payoffs.

Nodes are of two types: **Decision nodes** represent the points in the game at which players make a decision, i.e., choose an action, or a strategy in general. As any other tree, a game tree has a root and it is useful to distinguish the root, which we will call the **initial node**, from the other decision nodes (it is represented by an open circle whereas all the other nodes are represented by closed circles). To each decision node, including the initial node, one, and only one, player label is attached, to indicate who moves at that particular decision node. The second type of nodes are called **terminal nodes** and at these nodes the game is over and nobody takes any action anymore. To each terminal node a **payoff** vector is appended.

From each decision node, one or more branches emanate, each branch representing an action that can be taken by the player who is to move at that node. Each such branch is labelled with the **action** that it represents. A branch either leads to another decision node or to a terminal node.

The last component that we have to talk about is the **information sets.** Information sets tell us what the players know when they are making a decision. They are collections of decision nodes of a player that cannot be distinguished from the perspective of that player. We can illustrate it using the bargaining example under the assumption that the seller, somehow, does not observe the buyer's offer before deciding whether to accept or reject it. We depict this informational assumption by connecting the two decision nodes of the seller with a dashed line (see Figure 2).

Notice that the actions available to the seller at the two nodes that are in the same information set must be the same, otherwise the seller would be able to distinguish between them by just looking at the actions available to her.

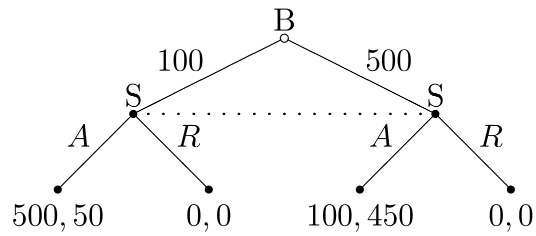

Figure 2: Bargaining Game with Imperfect Information

In this section we will deal with extensive form games with perfect information in which every player can distinguish between any two decision nodes and hence we will not have to worry about information sets.

1.2 Strategies

Strategies in a strategic form game are either action choices or probability distributions over actions. In an extensive form game, description of a strategy is more involved since players may have to choose actions at several points in the game. Therefore, a **pure strategy** of a player in an extensive form game has to specify an action choice at every decision node of that player. In that sense, a strategy is a complete plan of action, so complete that if it was handed over to a computer, the computer would know what to do under every contingency. We denote a pure strategy of player i by s_i, and the set of all pure strategies by S_i.

For example, in the extensive form game in Figure 1, a pure strategy for the buyer is easy enough: it has to specify what price to offer at the initial node. A pure strategy for the seller, on the other hand, has to specify an action at each decision node she may be called upon to move. So, the buyer has two pure strategies available to her: 100 and 500, and hence $S_B = \{100, 500\}$. The seller, however, has four pure strategies: (1) AA, (2) AR, (3) RA, (4) RR, and hence $S_S = \{AA, AR, RA, RR\}$.

The extensive form strategies sometimes lead to confusion. Let us try to illustrate why, by looking at the extensive form game in Figure 3.

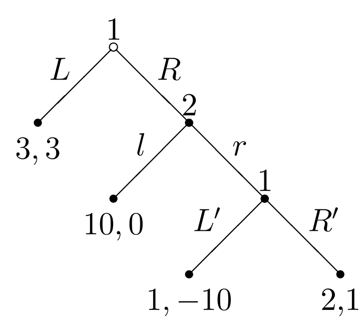

Figure 3: Another Extensive Form Game

A strategy for player 1 in this game has to specify an action at every decision node she has, and there are two such nodes. She, therefore, has four strategies: LL', LR', RL', RR'. Notice that the first two strategies specify an action even at player 1's second decision node which would not be reached if those strategies were implemented. The reason why, will become clear in the next section, after we analyze the optimal behavior of players. For now, let us look at the game tree of the senate-race game (see Figure 4) to further illustrate the concepts introduced so far.

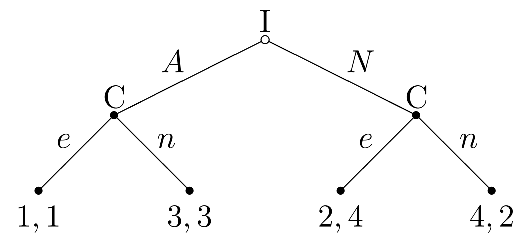

Figure 4: Senate-Race Game

In this game $S_I = \{A, N\}$ and $S_C = \{ee, en, ne, nn\}$.

2 Backward Induction Equilibrium

As in the strategic form games, the equilibrium concept in extensive form games is based upon the idea that each player plays a best response to the play of the other players. The difference is that we now require strategies to be optimal at every step in the game. The **backward induction equilibrium** is an algorithm that results in a recommendation of an action choice at every decision node with the property that if every player follows those recommendations their strategies would be optimal at every decision node they *may* be called upon to move. This will also result in a path of play (i.e., a sequence of branches) which will be called the **backward induction outcome**.

The algorithm is really simple. You, the game theorist, go to the final decision nodes and determine the best action available to the players who are to move at those nodes. Since there is no more moves after players make their moves at these decision nodes, this boils down to choosing the action that leads to the highest payoff for the player who is moving. (If there is a tie between two actions that lead to the highest payoff, you may simply choose one of them.) After you have done that, you prune all the actions that are not chosen (or just indicate the ones that are chosen by an arrow-head) and go to the penultimate decision nodes to determine the optimal action at those nodes. You continue in this manner until you reach to the initial node and determine the optimal action there.

For example, in the bargaining game we start with the seller's decision nodes which are the final decision nodes in the game tree. Since accepting both offers is optimal we mark the branches labelled A by arrow-heads. Once we do that, it is easily seen that the best action for player 1 is to offer \$100. Therefore, the backward induction equilibrium of the bargaining game is $(100, AA)$ and the backward induction outcome is $(100, A)$. (See Figure 5) The backward induction equilibrium of the senate race game is (A, ne) and its backward induction outcome is (A, n). (See Figure 6).

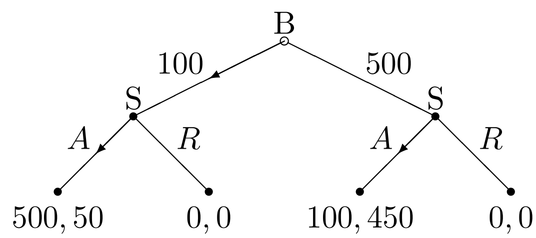

Figure 5: Bargaining Game

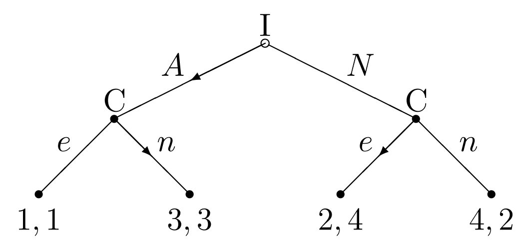

Figure 6: Senate-Race Game

As an exercise verify that the backward induction equilibrium of the game in Figure 3 is (LR', r). This example illustrates why player 1's strategy had to specify an action even after she has previously chosen L. Whether L is optimal or not for player 1 depends on what she believes that player 2 will do. If she believes that player 2 is going to choose l, then L is not optimal. But, whether player 2 will choose l or not depends on what player 2 believes that player 1 is going to do in her last decision node. Therefore, to determine the optimal action for player 1 at her first decision node, we have to specify what she intends to do at her last decision node.

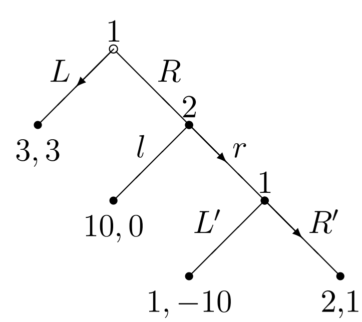

Figure 7: Another Extensive Form Game

2.1 Commitment and Mover Advantages

The bargaining and the senate-race games illustrate an important phenomenon that arise in many extensive form games, i.e., **the power of commitment**. Suppose that the seller could, somehow, commit herself to accepting only the offer 500 and that this is known to the buyer. Now, given that knowledge, the best that the buyer can do is to actually offer 500, because otherwise her offer will be rejected and she will receive 0, whereas offering 500 gives her 100. Therefore, public, and credible, commitments could increase a player's payoff in an extensive form game. Notice that this is similar to eliminating action A after the offer 100. This is in stark contrast to the single individual decision making problems where eliminating an action can never improve one's payoff.

Similarly, in the senate-race game, if the challenger could publicly commit to entering the race irrespective of the campaign decision of the incumbent, the best thing the incumbent could do would be not to run campaign ads and hence the challenger would respond by entering the race and obtaining a payoff of 4 rather than 3 that she was getting in the backward induction equilibrium.[1]

Another interesting phenomenon that arise in certain extensive form games is that of **first mover advantage**. For example, in the senate-race game, when the incumbent moves first, both players obtain a payoff of 3 in the backward induction equilibrium. Now, let us change the order of the moves so that it is the challenger who moves first so that we obtain the game tree depicted in Figure 8.

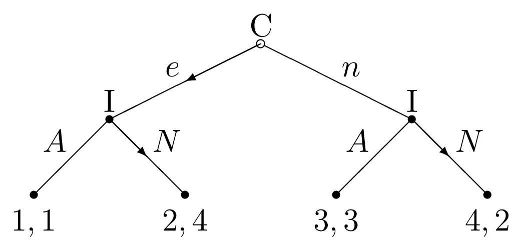

Figure 8: Modified Senate-Race Game

The backward induction equilibrium of this game is (e, NN) which yields a payoff of 4 to the challenger and a payoff of 2 to the incumbent. Therefore, if they had the chance, both players would prefer to move first in this game. This is similar to the idea behind the power of commitment. By choosing e the challenger commits herself to entering whatever the incumbent does.

However, not all games have a first mover advantage. Quite to the contrary, some games have **second mover advantage**. Consider a game in which the incumbent (who belongs

[1]See Thomas Schelling (1960), *The Strategy of Conflict*, for an excellent account of the idea of credible commitments.

to a rightist party) and the challenger (who belongs to a leftist party) in a senate race are choosing political platforms; either a leftist or a rightist one. Suppose that if both of them choose the same platform the incumbent wins the elections, whereas if they choose different platforms it is the challenger who wins. The candidates mostly care about winning, but they also would like to win (or lose) without compromising their political views. The game tree in Figure 9 depicts the situation if it is the incumbent who moves first, whereas the one in Figure 10 reverses the order of moves. Verify that this game exhibits second mover advantage.

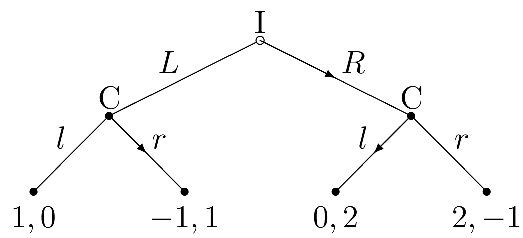

Figure 9: Senate-Race Game II

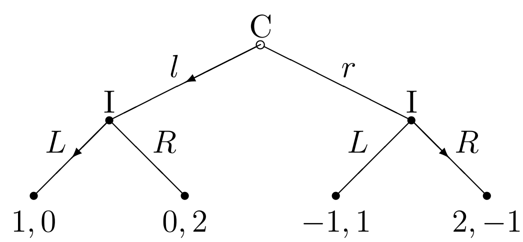

Figure 10: Modified Senate-Race Game II

3 Game Trees: A More Formal Treatment

A game tree is a collection of nodes, called T, and a binary relation between the nodes called a *precedence* relation, denoted $\succ$. Given two nodes α and β in the game tree, $\alpha \succ \beta$ means that α precedes β. Using this relation, we can define the set of predecessors of α as

$$P(\alpha) = \{t \in T : t \succ \alpha\}$$

and the set of successors as

$$S(\alpha) = \{t \in T : \alpha \succ t\}.$$

The set $P(\alpha)$ is simply the set of nodes from which one can go (through a sequence of branches) to α. Similarly, the set of successors of α is the set of nodes to which one can go starting from α.

The precedence relation $\succ$

1. is asymmetric, i.e., there exists no $\alpha, \beta \in T$ such that $\alpha \succ \beta$ and $\beta \succ \alpha$;

2. is transitive, i.e., $\alpha \succ \beta$ and $\beta \succ \gamma$ implies $\alpha \succ \gamma$;

3. there is a common predecessor to any two non-initial nodes, i.e., for all $\alpha, \beta \in T$, with $P(\alpha) \neq \emptyset$ and $P(\beta) \neq \emptyset$, there exists a $\gamma \in T$ such that $\gamma \in P(\alpha)$ and $\gamma \in P(\beta)$.

4. and satisfies the following property

$$\text{If } \alpha \succ \gamma \text{ and } \beta \succ \gamma, \text{ then either } \alpha \succ \beta \text{ or } \beta \succ \alpha.$$

The first two conditions guarantee that there are no cycles in the game tree, while the third condition guarantees that there is a unique initial node. The last condition guarantees that starting from any node there is a unique path back to the initial node.

Theorem 1 *Kuhn's (Zermelo's Theorem). Every finite extensive form game with perfect information has a backward induction equilibrium.*

Proof. Omitted.

4 Strategic Form of an Extensive Form Game

The strategic form is given by

1. The set of players N,

   and for each player i

2. The set of strategies S_i,

3. The payoff function,

$$u_i : S \rightarrow \mathbf{R}$$

   where $S = \times_{i \in N} S_i$ is the set of all strategy profiles.

So, the only difference from the standard definition of a strategic form game is the use of strategies rather than actions.

As an illustration, let us find the strategic form of the bargaining game. The set of players is $N = \{B, S\}$, the set of strategies are $S_B = \{100, 500\}$ and $S_S = \{AA, AR, RA, RR\}$. The payoff functions are represented in the following bimatrix

	AA	AR	RA	RR
100	$500, 50$	$500, 50$	$0, 0$	$0, 0$
500	$100, 450$	$0, 0$	$100, 450$	$0, 0$

Definition 2 *A strategy profile $s^* \in S$ is a Nash equilibrium if for each player i*

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \text{ for all } s_i \in S_i$$

or equivalently, if for each player i

$$s_i^* \in B_i\left(s_{-i}^*\right).$$

Therefore, the above bargaining game has three Nash equilibria $(100, AA)$, $(100, AR)$, and $(500, RA)$. Notice that the first two Nash equilibria result in the same outcome as does the backward induction equilibrium, i.e., $(100, A)$, whereas the third one results in the outcome $(500, A)$. This last equilibrium, however, is sustained by an **incredible threat** by the seller, i.e., the threat that she will not accept the offer of \$100. This threat is not credible because, if it was tested by the buyer, i.e., the buyer were to offer \$100, then the seller would actually find it in her interest to accept the offer.

Backward induction equilibrium concept eliminates equilibria based upon incredible threats by demanding players to be rational at every point in the game, a property that we call **sequential rationality**. Sequential rationality is stronger than just requiring the strategies to be best responses to the strategies of the other players, i.e., stronger than the rationality requirement behind the Nash equilibrium concept. For example, in the bargaining game above, the strategy RA is a best response to the offer of \$500, but is not sequentially rational, because it specifies the seller to reject the offer of \$100, and this is not rational at the decision node of the seller following the offer of \$100.

5 Extensive Form Games with Imperfect Information

In the previous section we have analyzed extensive form games with perfect information where every player had a perfect knowledge of what had happened previously in the game, i.e., each player observed the previous moves made by the other players. In this section we will relax this assumption and allow the possibility that some of the previous moves by other players are not observed when a player is called upon to move. Such games are called **extensive form games with imperfect information**.

In extensive form games with imperfect information, the notion of information sets, which we have introduced in the last section becomes crucial. An **information set** of player i is a collection of decision nodes of player i that cannot be distinguished by player i. Therefore, if the game reaches to any of the nodes in an information set of a player, that player does not know which of the nodes in that information node has actually been reached.

As an example consider the bargaining game with imperfect information (see Figure 2). In this game there is one information set of the seller that contains the decision nodes following the offers 100 and 500. When the seller is called upon to move, she does not know which of the two offers have been made, i.e., which of the two decision nodes in the information set has been reached. The strategy sets are given by $S_B = \{100, 500\}$ and $S_S = \{A, R\}$ and hence we have the following strategic form of this game

	A	R
100	$500, 50$	$0, 0$
500	$100, 450$	$0, 0$

The unique Nash equilibrium of this game is therefore $(100, A)$, the same outcome as the backward induction equilibrium outcome of the bargaining game with perfect information! The reason why we have a unique Nash equilibrium outcome in this game is that we have eliminated the seller's ability of making a non-credible threat of rejecting the offer of $100.

We may think of extensive form with imperfect information as a generalization of extensive form with perfect information. In the latter, all the information sets are singletons, i.e., they each contain a single node, whereas in the former there is at least one information set that contains more than one node.

As an another example consider the following entry-game. Suppose Pepsi is currently the sole provider in a market, say in Bulgaria. Coke is considering to enter the market. If Coke enters, both firms simultaneously decide whether to act tough (T) or accommodate (A). This leads to an extensive form game with imperfect information whose game tree representation is given in Figure 11, where the first number in a payoff vector belongs to Coke and the second to Pepsi.

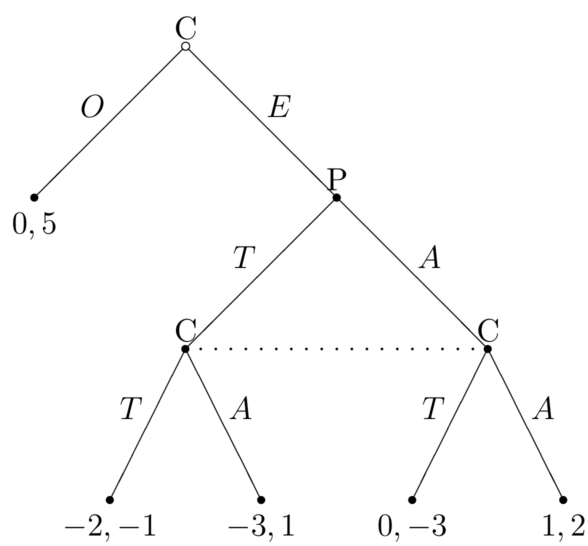

Figure 11: Entry-game.

In this game $S_C = \{OT, OA, ET, EA\}$ and $S_P = \{T, A\}$, and hence we have the following strategic form:

	T	A
OT	$0,5$	$0,5$
OA	$0,5$	$0,5$
ET	$-2,-1$	$0,-3$
EA	$-3,1$	$1,2$

There are three Nash equilibria of this game: (OT,T), (OA,T), (EA,A). In the second Nash equilibrium Coke is supposed to accommodate and Pepsi is supposed to act tough, following Coke entering the market. Is that reasonable? In other words, suppose, the game actually reached that stage, that is Coke actually entered. Now, is (A,T) a reasonable outcome? One way of asking the same question is to check if both players are acting rationally, i.e., best responding to each other's strategies, conditional upon Coke entering the market. Notice that conditional upon Coke entering the market we have the following "game"

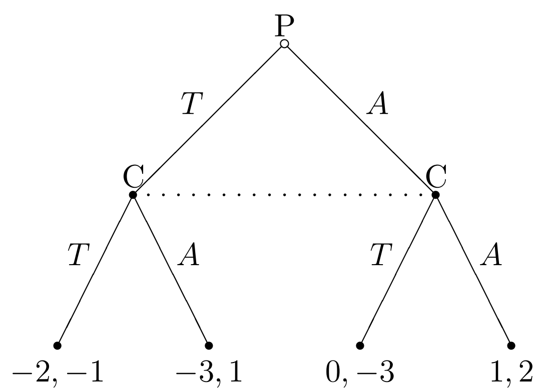

Figure 12: Entry-"game".

which has the following strategic form

		Pepsi T	Pepsi A
Coke	T	$-2,-1$	$0,-3$
	A	$-3,1$	$1,2$

If Coke anticipates Pepsi to play T, then its best response is T as well, not A. (Neither is T a best response for Pepsi to A.) Therefore, to the extent that we regard only Nash equilibrium outcomes as reasonable, we conclude that (A,T) is not reasonable. In contrast, the post-entry behavior of both players are rational in equilibria (OT,T) and (EA,A).

5.1 Subgames and Subgame Perfect Equilibrium

Subgame perfect equilibrium is a generalization of the backward induction equilibrium to extensive form games with imperfect information. To define subgame perfect equilibrium we have to first define a subgame.

Definition 3 *A* ***subgame*** *is a part of the game tree such that*

1. *it starts at a single decision node,*
2. *it contains every successor to this node,*
3. *if it contains a node in an information set, then it contains all the nodes in that information set.*

It is conventional to treat the entire game as a subgame and call all the other subgames **proper subgames.** For example, the entry-game given in figure 11 has two subgames: the game itself and the subgame which starts after Coke enters the market. Of course, only the latter is a proper subgame.

Given a subgame g, let us denote the restriction of a strategy s_i to that subgame g by $s_i|_g$. For example, if we denote the post-entry subgame in the entry-game by e (this subgame is given in figure 12), then $OT|_e = T$, $EA|_e = A$, etc.

Definition 4 *A strategy profile s^* in an extensive form game Γ is a* ***subgame perfect equilibrium*** *(SPE) if for every subgame g of Γ, $s^*|_g$ is a Nash equilibrium of g.*

Therefore, there are two SPE of the entry-game: (OT, T) and (EA, A).

We can now obtain a better insight into the difference between subgame perfect equilibrium (or backward induction equilibrium) and Nash equilibrium by using the language of subgames. We first have to distinguish between subgames that can be reached by a strategy profile and those that cannot be reached. A subgame can be **reached** under the strategy profile $s \in S$ if, when the strategy profile is implemented, the initial node of the subgame will actually be reached. Otherwise, we say that the subgame cannot be reached under the strategy profile s. A strategy profile s^* is a Nash equilibrium if every player plays a best response to the strategies of the other players in every subgame that can be reached under s^*. In contrast, a strategy profile s^* is a SPE if every player plays a best response to the strategies of the other players in every subgame, i.e., even in those subgames that cannot be reached under s^*. In other words, Nash equilibrium demands rationality in only those subgames that can be reached in equilibrium, whereas SPE demands rationality in every subgame, and this latter form of rationality is called **sequential rationality**.

As an exercise consider the game in figure 3 and find its Nash equilibria and SPE. Verify that there are Nash equilibria in which one of the players do not behave sequentially rationally, whereas in all SPE both players act sequentially rationally.