1. Find the Nash and subgame perfect equilibria of the games given in Figure 1 and Figure 2.

Solution

Entry Game

Strategic form of the game is given by

Table:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>OF</td>
<td>0, 20</td>
<td>0, 20</td>
</tr>
<tr>
<td>OA</td>
<td>0, 20</td>
<td>0, 20</td>
</tr>
<tr>
<td>EF</td>
<td>-5, -5</td>
<td>15, 5</td>
</tr>
<tr>
<td>EA</td>
<td>5, 15</td>
<td>10, 10</td>
</tr>
</tbody>
</table>

Note that OF and OA are strictly dominated and hence receive zero probability in any Nash equilibrium. We can then easily verify that the set of Nash equilibria in mixed strategies are given by

\( (\sigma_1(EF) = 1, \sigma_2(A) = 1), (\sigma_1(EA) = 1, \sigma_2(F) = 1), (\sigma_1(EA) = 1/3, \sigma_2(A) = 1/3) \)

Since these induce Nash equilibria in the only proper subgame of the game they are also subgame perfect equilibria.

Selten’s Horse

Let’s organize search for equilibria according to player 1’s strategy. Since each player has only one information set, there is no possibility of confusion if we write \( \beta_i(a) \) for \( \beta_i(a|I) \). There are three possibilities.
2. Currently, in the market for very large aircrafts (VLA) there is only one product, a Boeing 747. Airbus each outcome separately.

(ii) Only Boeing has SJ; (iii) Only Airbus has SJ; (iv) Both Airbus and Boeing have SJ. Let us analyze

the marginal cost of a 747 and an SJ is

c > 0. Assume for simplicity that development of SJ is costless, the
costs
e of a 747 at price p is v(0, x) = u – tx² – p. Therefore, she will buy only if p ≤ u and

\[ x \leq \sqrt{\frac{u - p}{t}} \]

Therefore, assuming that \( u - t < p < u \), Boeing’s profit is given by

\[ \pi(p) = (p - c) \sqrt{\frac{u - p}{t}} \]

and profit maximization implies

\[ \sqrt{\frac{1}{t} (u - p)} - \frac{1}{2t} \frac{p - c}{\sqrt{\frac{1}{t} (u - p)}} = 0 \]

Since there is no proper subgame of the game, the above also are subgame perfect equilibria.

2. Currently, in the market for very large aircrafts (VLA) there is only one product, a Boeing 747. Airbus (A) and Boeing (B) are considering to develop another VLA, a super jumbo (SJ). The game is composed of three stages:

1. First Boeing and, after observing Boing’s decision, Airbus decides whether to develop (D) an SJ or not (N);
2. If already developed an SJ, Airbus and Boeing simultaneously decide whether to stay (S) in the SJ market or not (E);
3. Remaining firms simultaneously decide on prices of 747 and SJ.

We may characterize a VLA by a number between 0 and 1, so that a higher number corresponds to a larger plane. A 747 is located at 0 and an SJ at 1. There is a continuum of customers distributed uniformly between 0 and 1, where the location of a customer stands for her ideal aircraft size. Customer with ideal point \( x \in [0, 1] \) who purchases a jet with attribute \( s \in [0, 1] \) at price p receives a payoff equal to \( v(s, x) = u - t(s - x)^2 - p \), where u, t > 0. Assume that each customer will buy at most one jet and the payoff to not buying a jet is zero. Also assume for simplicity that development of SJ is costless, the marginal cost of a 747 and an SJ is c > 0, where \( c + 1.25t < u < c + 3t \), and that exiting the SJ market costs \( e \in (0, 3t/8) \). Find the subgame perfect equilibria of this game.

Solution

As Figure 3 illustrates there are four possible market outcomes: (I) Neither Boeing nor Airbus has SJ; (II) Only Boeing has SJ; (III) Only Airbus has SJ; (IV) Both Airbus and Boeing have SJ. Let us analyze each outcome separately.

I. Neither Boeing nor Airbus has SJ

In this case Boeing is a monopolist in the VLA market with its 747. Payoff of customer x if she buys 747 at price p is \( v(0, x) = u - tx^2 - p \). Therefore, she will buy only if \( p \leq u \) and

\[ x \leq \sqrt{\frac{u - p}{t}} \]

2. If already developed an SJ, Airbus and Boeing simultaneously decide whether to stay (S) in the SJ market or not (E);

3. Remaining firms simultaneously decide on prices of 747 and SJ.

(c) \( \beta_1(D) \in (0, 1) \). First assume that \( \beta_2(c) = 1 \). For player 1 to completely mix, we then need \( \beta_3(L) = 1/3 \), to which \( \beta_2(c) = 1 \) is not a best response. Second, assume \( \beta_2(d) = 1 \). For player 1 to completely mix, we need \( \beta_3(L) = 0 \), to which \( \beta_2(d) = 1 \) is not a best response. Third, assume \( \beta_2(c) \in (0, 1) \). For this to be a best response we need \( \beta_3(L) = 1/4 \), but then playing C is strictly better for player 1. Therefore, there is no Nash equilibrium in which \( \beta_1(D) \in (0, 1) \).

Since there is no proper subgame of the game, the above also are subgame perfect equilibria.
Therefore, if the optimal price satisfies $u - t < p < u$, we must have

$$p = \frac{2u + c}{3}$$

At this price profit is equal to

$$\frac{2(u - c)^{3/2}}{3\sqrt{3t}}$$

If Boeing sets the price $p \geq u$, its profit is equal to

$$0 < \frac{2(u - c)^{3/2}}{3\sqrt{3t}}$$

So, optimal price must be smaller than $u$. If, on the other hand, $p < u - t$, then everybody buys and Boeing’s profits is at most $u - t - c$. Our assumption that $u < c + 3t$ implies that

$$u - t - c < \frac{2(u - c)^{3/2}}{3\sqrt{3t}}$$

and hence the optimal price is indeed given by

$$p = \frac{2u + c}{3}$$

II. Only Boeing has SJ

We first show that the market is covered, that is each customer either buys a 747 or a SJ. To do this, suppose to the contrary that there is a group of customers who buy neither. Then, there must exist $x_1$ and $x_2$ such that $x_1 < x_2$ and any customer whose ideal size is in between $x_1$ and $x_2$ buys no aircraft. This is the case because if a customer finds it profitable to buy a 747, then every customer whose ideal size is smaller, i.e., closer to 0, will also find it profitable to buy a 747. Similarly, if a customer finds it profitable to buy a SJ, then every customer whose ideal size is closer to 1 will also find it profitable to buy a SJ.

Therefore, the customer whose ideal size is $x_1$ is indifferent between buying a 747 and not buying at all and the customer with ideal size $x_2$ is indifferent between buying a SJ and not buying any aircraft. In other words, if Boeing’s price for 747 is $p_0$ and for the SJ is $p_1$ the following must be true:

$$u - tx_1^2 - p_0 = 0$$
$$u - t(1 - x_2)^2 - p_1 = 0$$

Second thing to note is that Boeing must be acting as if it is a monopolist in two separate markets. This is because a small change in the price of a 747 only produces a small change in demand for 747 and does not affect the demand for a SJ. Similarly for a small change in the price of the SJ.
Let’s first determine what should be the optimal price of the 747. Solving the first equation above for \( x_1 \) and assuming \( p_0 \leq u \) we get

\[
    x_1 = \sqrt{\frac{u - p_0}{t}}
\]

Therefore, Boeing’s profits from 747 is given by

\[
    (p_0 - c) \sqrt{\frac{u - p_0}{t}}
\]

if \( p_0 < u \) and is equal to zero if \( p_0 \geq u \). Assume that the optimal price \( p_0 < u \). Then the first order condition must hold:

\[
    \sqrt{\frac{1}{t} (u - p_0)} - \frac{1}{2t} \frac{p_0 - c}{\sqrt{\frac{1}{t} (u - p_0)}} = 0
\]

Solving for \( p_0 \) we get

\[
    p_0 = \frac{2u + c}{3}
\]

This results in positive profits, whereas setting \( p_0 \geq u \) would result in zero profit. Therefore, optimal price is indeed

\[
    p_0 = \frac{2u + c}{3}
\]

and

\[
    x_1 = \sqrt{\frac{u - c}{3t}}
\]

Our assumption that \( u > c + 1.25t \) implies

\[
    x_1 > \sqrt{\frac{c + 1.25t - c}{3t}} = \sqrt{\frac{1.25}{3}} > \frac{1}{2}
\]

A similar analysis of optimal \( p_1 \) shows that \( x_2 < 1/2 \) which contradicts our hypothesis that \( x_1 < x_2 \). Therefore, the market must be covered, i.e., Boeing sells an aircraft to every customer. Therefore, the market must be split between 747 and the SJ. Let \( x^* \) be the customer who is indifferent between buying a 747 and a SJ:

\[
    u - t(0 - x^*)^2 - p_0 = u - t(1 - x^*)^2 - p_A
\]

Therefore,

\[
    x^* = \frac{p_1 - p_0 + t}{2t}
\]

If \( x^* \in (0, 1) \), i.e., Boeing sells both aircrafts, its profit function therefore is given by

\[
    (p_0 - c)x^* + (p_1 - c)(1 - x^*)
\]

or after substituting for \( x^* \)

\[
    (p_0 - c)\frac{p_1 - p_0 + t}{2t} + (p_1 - c)\frac{p_0 - p_1 + t}{2t}
\]

Collecting terms we can write the profit function as

\[
    \frac{1}{2t} \left[ t(p_0 + p_1) - (p_1 - p_0)^2 - 2tc \right]
\]

It is best for Boeing to set \( p_0 = p_1 \) because profit function is decreasing in \((p_1 - p_0)\). Since \( p_0 = p_1 \) the customer who is indifferent between 747 and the SJ is given by \( x^* = 1/2 \). From above we know that the market is covered, i.e., customer \( x^* = 1/2 \) must be willing to buy a 747 or a SJ:

\[
    u - t(1/2)^2 - p_0 \geq 0
\]

\[
    u - t(1/2)^2 - p_1 \geq 0
\]

In other words, it must be the case that

\[
    p_0 \leq u - 0.25t \quad p_1 \leq u - 0.25t
\]
As long as these conditions hold Boeing sells to the entire customer base. Therefore, Boeing would set the highest prices that satisfy these conditions, i.e.,

\[ p_0 = p_1 = u - 0.25t \]

and its profit is \( u - 0.25t - c \)

If Boeing were to sell only 747 or SJ its highest profit would be

\[
\frac{2(u-c)^{3/2}}{3\sqrt{3}t} < u - 0.25t - c
\]

by our assumption that \( c + 1.25t < u \). Thus selling both is indeed the optimal policy.

III. Only Airbus has SJ

This is the same model as the Linear City model analyzed in handout “Strategic Form Games: Applications.” The equilibrium prices for Airbus and Boeing are \( p_A = p_B = c + t \), which gives a profit of \( t/2 \) to each.

IV. Both Airbus and Boeing have SJ

In this case we have Bertrand model with homogenous products in the SJ market which drives prices to \( c \). Again following the analysis on Linear City model, we can show that Boeing will set its price for 747 \( p_0 \) to maximize

\[
(p_0 - c)\frac{c - p_0 + t}{2t}, \quad \text{s.t. } p_0 \in [c - t, c + t]
\]

The solution to this problem is \( p_0 = c + 0.5t \). Therefore, Airbus gets a profit of zero in this case and Boeing’s profit is \( t/8 \).

Table 1 gives the profits corresponding to different outcomes:

<table>
<thead>
<tr>
<th></th>
<th>Boeing</th>
<th>Airbus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Only B has SJ</td>
<td>( u - 0.25t - c )</td>
<td>0</td>
</tr>
<tr>
<td>Neither has SJ</td>
<td>( \frac{2(u-c)^{3/2}}{3\sqrt{3}t} )</td>
<td>0</td>
</tr>
<tr>
<td>Only A has SJ</td>
<td>( t/2 )</td>
<td>( t/2 )</td>
</tr>
<tr>
<td>Both have SJ</td>
<td>( t/8 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Profits

Our assumptions imply that

\[ u - 0.25t - c > \frac{2(u-c)^{3/2}}{3\sqrt{3}t} > t/2 > t/8 \]

Therefore, we have the reduced subgame given in Figure 4, where the first number is Boeing’s payoff.

In the subgame played after both develop SJ’s, Stay is dominant for Airbus and, since \( e < 3t/8 \), Boeing’s best response is Exit. Therefore, in the unique Nash equilibrium Boeing exits and Airbus stays.

Moving backward we find that developing SJ is a dominant strategy for Airbus and therefore Boeing should not develop. This implies that in the unique SPE outcome Airbus develops SJ and Boeing does not.

3. Consider the game in Figure 1 and find the behavioral strategy equivalent to the mixed strategy given by \( \sigma_1(OF) = 0.2, \sigma_1(OA) = 0.3, \sigma_1(EF) = 0.5 \)

Solution

For any \( I \in \mathcal{I}_i \) and \( a \in A(I) \) we define an outcome equivalent \( \beta_i \) as follows:

\[
\beta_i(a|I) = \begin{cases} 
\frac{\sum_{s_i \in S_i(I) \cap S_i(a)} \sigma_i(s_i)}{\sum_{s_i \in S_i(I)} \sigma_i(s_i)}, & \text{if } I \in T^x_i \\
\sum_{s_i \in S_i(a)} \sigma_i(s_i), & \text{if } I \notin T^x_i
\end{cases}
\]
In this case $S_1 = \{OF, OA, EF, EA\}$ and $\text{supp}(\sigma_1) = \{OF, OA, EF\}$. Therefore, $T_1^{\sigma_1} = \{(a_0), \{(E, F), (E, A)\}\}$. Since $\{a_0\} \in T_1^{\sigma_1}$

$$\beta_1(O|\{a_0\}) = \frac{\sum_{s_1 \in S_1(\{a_0\}) \cap S_1(O)} \sigma_1(s_1)}{\sum_{s_1 \in S_1(\{a_0\})} \sigma_1(s_1)}$$

Since $S_1(\{a_0\}) = \{OF, OA, EF, EA\}$ and $S_1(O) = \{OF, OA\}$

$$\beta_1(O|\{a_0\}) = \frac{\sigma_1(OF) + \sigma_1(OA)}{\sigma_1(OF) + \sigma_1(OA) + \sigma_1(EF) + \sigma_1(EA)} = \frac{1}{2}$$

Similarly, $\{(E, F), (E, A)\} \in T_1^{\sigma_1}$ and $S_1(\{(E, F), (E, A)\}) = \{EF, EA\}$ and $S_1(F) = \{OF, EF\}$

$$\beta_1(F|\{(E, F), (E, A)\}) = \frac{\sigma_1(EF)}{\sigma_1(EF) + \sigma_1(EA)} = 1$$