Extensive Form Games

An extensive form game is defined by

- $N$: a finite set of players
- $H$: a set of histories (or nodes) such that
  1. $a_0 \in H$ (called the initial node or empty history)
  2. For any integer $k \geq 0$, $(a_0, a_1, \ldots, a_k) \in H$ whenever $(a_0, a_1, \ldots, a_{k+1}) \in H$
  3. If $(a_k)_{k=0}^{\infty}$ satisfies $(a_k)_{k=0}^{K} \in H$ for every integer $K$, then $(a_k)_{k=0}^{\infty} \in H$
- $\pi: H \setminus Z \to N \cup \{c\}$: the player function
  - $\pi(h) = i$: $i$ moves immediately after history $h$ and chooses an action from the set $A(h)$
  - If $\pi(h) = c$, then chance determines the action taken after history $h$
- Set of histories after which $i \in N \cup \{c\}$ moves: $H_i = \{h \in H: \pi(h) = i\}$
- Set of actions available to $i \in N \cup \{c\}$ in the entire game: $A_i = \bigcup_{\pi(h) = i} A(h)$
- $\beta_c$: chance’s probability distribution
  - For any $h \in H_c$, $\beta_c(\cdot|h)$ is a probability distribution over $A(h)$
  - $\beta_c(a|h)$ denotes the probability assigned to $a \in A(h)$
  - We sometimes use “nature” instead of chance

Extensive Form

- The strategies in strategic form games are specified so that each player chooses an action (or a mixture of actions) once and for all.
- In games that involve a sequential character this specification may not be suitable as players may find it profitable to reassess their strategies as the events in the game unfold.
- Extensive form provides an explicit description of a strategic interaction by specifying
  - who moves when
  - to do what
  - with what information
- It provides a richer environment to study interesting questions such as
  - commitment
  - repeated interaction
  - signaling
  - reputation building, etc.
- A given game can be represented in both a strategic form and an extensive form
  - The choice depends on the questions that we ask
Extensive Form Games

An information partition \( \mathcal{I} \) of \( H \setminus Z \) such that for all \( I \in \mathcal{I} \) and all \( h, h' \in I \) we have

- Each element of \( \mathcal{I} \) is an information set
- The information set containing \( h \) is denoted \( I(h) \)
- Set of information sets of \( i \in N \cup \{c\} \)

\[ \mathcal{I}_i = \{I(h) : i(h) = i, \text{for some } h \in H \setminus Z\} \]

We may write \( A(I) \) instead of \( A(h) \), \( i(I) \) instead of \( i(h) \), and \( \beta_c(a|I) \) instead of \( \beta_c(a|h) \) for any \( h \in I \)

For each \( i \in N \), a von Neumann-Morgenstern payoff function

\[ u_i : Z \rightarrow \mathbb{R} \]

The Dating Game

- We can use game trees to represent extensive form games
- This is a game with incomplete information

\[ N = \{1, 2\} \]
\[ H = \{a_0, S, D, (S, f), (D, f), (S, q), (D, q), (S, f, m), (S, f, d), (D, f, m), (D, f, d), (S, q, m), (S, q, d), (D, q, m), (D, q, d)\} \]
\[ Z = \{(S, f, m), (S, f, d), (D, f, m), (D, f, d), (S, q, m), (S, q, d), (D, q, m), (D, q, d)\} \]

We may omit \( a_0 \) from the description of any other history

\[ |(S, f, d)| = 3 \]
\[ \iota(a_0) = c, \iota(S) = \iota(D) = 1, \iota(S, f) = \iota(D, f) = \iota(S, q) = \iota(D, q) = 2 \]
\[ A(a_0) = \{S, D\}, A(S) = A(D) = \{f, q\} \]
\[ A(S, f) = A(D, f) = A(S, q) = A(D, q) = \{m, d\} \]
Sequential Battle of the Sexes

- This is a game with perfect information

\[ \beta_c(S|a_0) = 1/3, \beta_c(D|a_0) = 2/3 \]
\[ I = \{(a_0), \{S\}, \{D\}, \{(S, f), (D, f)\}, \{(S, q), (D, q)\}\} \]
\[ u_1(S, f, m) = 1, u_1(S, f, d) = -1, \text{ etc.} \]
\[ u_2(S, f, m) = 1, u_2(S, f, d) = 0, \text{ etc.} \]

Sequential Battle of the Sexes: A Game with Perfect Information

\[ N = \{1, 2\} \]
\[ H = \{a_0, x, (x, B), (x, S), (x, B, B), (x, B, S), (x, S, B), (x, S, S)\} \]
\[ Z = \{(x, B, B), (x, B, S), (x, S, B), (x, S, S)\} \]
\[ \iota(a_0) = c, \iota(x) = 1, \iota(x, B) = \iota(x, S) = 2 \]
\[ A(a_0) = \{x\}, A(x) = \{B, S\}, A(x, B) = A(x, S) = \{B, S\} \]
\[ \beta_c(x|a_0) = 1 \]
\[ I = \{(a_0), \{x\}, \{(x, B)\}, \{(x, S)\}\} \]
\[ u_1(B, B) = 2, u_1(B, S) = 0, u_1(S, B) = 0, u_1(S, S) = 1 \]
Since the game is with perfect information we may omit information partition $I$ from the definition.

Since chance's role is trivial, we may omit anything about it from the definition.

$$N = \{1, 2\}$$

$$H = \{a_0, B, S, (B, B), (B, S), (S, B), (S, S)\}$$

$$Z = \{(B, B), (B, S), (S, B), (S, S)\}$$

$$\iota(a_0) = 1, \iota(B) = \iota(S) = 2$$

$$u_1((B, B)) = 2, u_1((B, S)) = 0, u_1((S, B)) = 0, u_1((S, S)) = 1$$

### Extensive Form Strategies

- **A pure strategy:** For any $i \in N \cup \{c\}$
  
  $$s_i : I_i \rightarrow A_i$$ with $s_i(I) \in A(I)$ for all $I \in I_i$

  The set of pure strategies for $i \in N \cup \{c\}$

  $$S_i = \times_{I \in I_i} A(I)$$

- **Set of pure strategy profiles:**

  $$S = \times_{i \in N} S_i$$

  Note that we also define pure strategies for Nature, although it is a non-strategic “player”. We do this only for notational convenience in defining outcomes and expected payoffs of players in $N$.

  Players may randomize in extensive form games. Later on, we will introduce different ways through which they may do so.

---

**A Game with Imperfect and Complete Information**

- **N** = \{1, 2\}
- **H** = \{a_0, B, S, (B, B), (B, S), (S, B), (S, S)\}
- **Z** = \{(B, B), (B, S), (S, B), (S, S)\}
- **$\iota(a_0)$** = 1, **$\iota(B)$** = **$\iota(S)$** = 2
- **$u_1((B, B))$** = 2, **$u_1((B, S))$** = 0, **$u_1((S, B))$** = 0, **$u_1((S, S))$** = 1

---

**Extensive Form Strategies**

- **Dating Game:**
  
  $$s_c : \{\{a_0\}\} \rightarrow \{S, D\}$$
  $$s_1 : \{\{S\}, \{D\}\} \rightarrow \{q, f\}$$
  $$s_2 : \{\{(S, f), (D, f)\}, \{(S, q), (D, q)\}\} \rightarrow \{m, d\}$$

- **Sequential Battles of the Sexes:**
  
  $$s_1 : \{\{a_0\}\} \rightarrow \{B, S\}$$
  $$s_2 : \{\{B\}, \{S\}\} \rightarrow \{B, S\}$$

  **Set of pure strategies:**

  $$S_1 : \{B, S\}$$

  $$S_2 : \{BB, BS, SB, SS\}$$
Extensive Form Strategies
Consider the following (centipede) game

\[
\begin{array}{cccc}
1 & C & 2 & C \\
S & S & S & \\
1, 0 & 0, 2 & 3, 1 & 2, 4
\end{array}
\]

- \( S_1 = \{SS, SC, CS, CC\} \)
- \( S_2 = \{S, C\} \)

An extensive form strategy has to specify how a player will act after any history she may be called upon to move, including those that are precluded by the previous play of the same player.

Extensive Form Games with Perfect Information
- Each player observes the previous moves of every player (including nature)
- Every information set is a singleton
- Consider the centipede game again

How do you think rational players will play this game?
We have just applied backward induction algorithm
Backward induction equilibrium of the centipede game is given by

\[ s_1(a_0) = S, s_1(C, C) = S, s_2(C) = S \]

We could write \( s_1 = SS, s_2 = C \) when the meaning is clear
Backward induction outcome is \( S \)

Extensive Form Strategies and Payoffs
- A pure strategy profile \( s \in S \) and Nature’s probability distribution \( \beta_c \) induces an outcome \( P^s \)
  - A probability distribution over the set of terminal histories \( Z \)
  - Let \( P^s(z) \) denote the probability of \( z \in Z \) under \( s \in S \)
- We assume that players’ preferences can be represented by expected payoffs:

\[
U_i(s) = \sum_{z \in Z} P^s(z) u_i(z)
\]

For example in sequential BoS game if \( s_1 = B \) and \( s_2(B) = S, s_2(S) = B \) then the outcome is \( (B, S) \) with probability 1:

\[ P^s(B, S) = 1 \]

In the dating game if \( s_1(S) = f, s_1(D) = q, s_2(\{(S, f), (D, f)\}) = m, s_2(\{(S, q), (D, q)\}) = d \), then

\[ P^s(S, f, m) = 1/3, P^s(D, q, d) = 2/3 \]

Next we will start discussing solution concepts for extensive form games. We begin with a conceptually simpler form: extensive form games with perfect information.

Backward Induction

Backward induction equilibrium:

\[ s_1(C, C) = S \]

\[
\begin{array}{cccc}
1 & C & 2 & C \\
S & S & S & \\
1, 0 & 0, 2 & 3, 1 & 2, 4
\end{array}
\]

\[ s_2(C) = S \]

\[
\begin{array}{cccc}
1 & C & 2 & C \\
S & S & S & \\
1, 0 & 0, 2 & 3, 1 & 2, 4
\end{array}
\]

\[ s_1(a_0) = S \]

\[
\begin{array}{cccc}
1 & C & 2 & 0.2 \\
S & S & S & \\
1, 0 & 0, 2 & 3, 1 & 2, 4
\end{array}
\]

Backward induction equilibrium: \( s_1 = SS, s_2 = S \)
### Nash Equilibrium

**Definition**

A (pure strategy) Nash equilibrium of an extensive form game with perfect information \( \Gamma = (N, H, \iota, \beta_c, I, (u_i)) \) is a strategy profile \( s^* \in S \) such that for every player \( i \in N \)

\[
U_i(s^*_i, s^*_{-i}) \geq U_i(s_i, s^*_{-i}) \quad \text{for all } s_i \in S_i
\]

Nash equilibrium of an extensive form game can also be defined as the Nash equilibrium of its strategic form.

**Definition**

The strategic form of the extensive form game with perfect information \( \Gamma = (N, H, \iota, \beta_c, I, (u_i)) \) is the strategic form game \( G_\Gamma = (N, (S_i), (U_i)) \).

---

### Nash Equilibrium: Centipede Game

**Strategic Form of the centipede game:**

- \( N = \{1, 2\} \)
- \( S_1 = \{SS, SC, CS, CC\}, S_2 = \{S, C\} \)
- Payoffs are given in the following bimatrix:

<table>
<thead>
<tr>
<th></th>
<th>( S )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SS )</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>( SC )</td>
<td>1,0</td>
<td>0,2</td>
</tr>
<tr>
<td>( CS )</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>( CC )</td>
<td>2,4</td>
<td>1,0</td>
</tr>
</tbody>
</table>

- \( \text{BI}(\Gamma) = \{(SS, S)\} \)
- \( \text{N}(\Gamma) = \{(SS, S)(SC, S)\} \)
- BIE and NE are outcome equivalent in this game
- This is not always the case

---

### Entry Game

**Strategic Form of the Game**

- \( (O, F) \) equilibrium sustained by an incredible threat
- Backward induction rules incredible threats or promises out
- NE requires rationality whereas BIE requires sequential rationality
  - Players must play optimally at every point in the game

---

### Proposition

Every finite extensive form game with perfect information has a backward induction equilibrium.

### Proposition

If \( s^* \) is a backward induction equilibrium of a finite extensive form game with perfect information \( \Gamma \), then \( s^* \) is a Nash equilibrium of \( \Gamma \).

### Proposition

Every finite extensive form game with perfect information has a Nash equilibrium.
Example: Stackelberg Duopoly

- Remember Cournot Duopoly model?
  - Two firms simultaneously choose output (or capacity) levels
  - What happens if one of them moves first?
    - or can commit to a capacity level?
- The resulting model is known as Stackelberg oligopoly
  - After the German economist Heinrich von Stackelberg in *Marktform und Gleichgewicht* (1934)
- The model is the same except that, now, Firm 1 moves first

Payoff function of each firm is given by

\[ u_i(q_1, q_2) = \begin{cases} 
(a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq \frac{a}{b} \\
-cq_i, & q_1 + q_2 > \frac{a}{b} 
\end{cases} \]

Nash Equilibrium of Cournot Duopoly

Best response correspondence of firm \( i \neq j \) is given by

\[ B_i(q_j) = \begin{cases} 
\frac{a-c-bq_j}{b}, & q_j < \frac{a-c}{b} \\
0, & q_j \geq \frac{a-c}{b} 
\end{cases} \]

Unique Nash equilibrium:

\[ q_1^* = q_2^* = \frac{a-c}{3b} \]

Equilibrium profits:

\[ \pi_1^* = \pi_2^* = \frac{(a-c)^2}{9b} \]

Stackelberg Model

The game has two stages:
- Firm 1 chooses a capacity level \( q_1 \geq 0 \)
- Firm 2 observes Firm 1’s choice and chooses a capacity \( q_2 \geq 0 \)
- \( N = \{1, 2\} \)
- \( H = \{a_0\} \cup \mathbb{R}_+ \cup \mathbb{R}_+^2 \)
- \( Z = \mathbb{R}_+^2 \)
- \( \iota(a_0) = 1, \iota(q_1) = 2 \), for all \( q_1 \in \mathbb{R}_+ \)
- \( u_1(q_1, q_2) = \begin{cases} 
(a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq \frac{a}{b} \\
-cq_i, & q_1 + q_2 > \frac{a}{b} 
\end{cases} \)
- \( q_1 \in \mathbb{R}_+ \)
- \( q_2 : \mathbb{R}_+ \to \mathbb{R}_+ \)

Backward Induction Equilibrium of Stackelberg Game

- Sequential rationality of Firm 2 implies that for any \( q_1 \) it must play a best response:

\[ q_2(q_1) = \begin{cases} 
\frac{a-c-bq_1}{2b}, & q_1 < \frac{a-c}{b} \\
0, & q_1 \geq \frac{a-c}{b} 
\end{cases} \]

- Firms 1’s strategy has to maximize:

\[ u_1(q_1, q_2(q_1)) = \begin{cases} 
(a - c - b(q_1 + q_2(q_1)))q_i, & q_1 + q_2(q_1) \leq \frac{a}{b} \\
-cq_i, & q_1 + q_2(q_1) > \frac{a}{b} 
\end{cases} \]

- Any \( q_1 \geq (a-c)/b \) leads to non-positive payoff whereas \( 0 < q_1 < (a-c)/b \) gives positive payoff

- Therefore, Firm 1 will choose \( q_1 \) to maximize

\[ [a - c - b(q_1 + \frac{a-c-bq_1}{2b})]q_i \]

- Since \( q_1 = 0 \) cannot be the solution, FOC must hold with equality

\[ q_1 = \frac{a-c}{2b} \]
Backward Induction Equilibrium of Stackelberg Game

Backward Induction Equilibrium

\[ q_1^* = \frac{a - c}{2b} \]
\[ q_2^*(q_1) = \begin{cases} \frac{a - c - bq_1}{2b} & q_1 < \frac{a - c}{b} \\ 0 & q_1 \geq \frac{a - c}{b} \end{cases} \]

Backward Induction Outcome

\[ q_1^s = \frac{a - c}{2b} > \frac{a - c}{3b} = q_1^e \]
Firm 1 commits to an aggressive strategy

\[ q_2^s = \frac{a - c}{4b} < \frac{a - c}{3b} = q_2^e \]

Equilibrium Profits

\[ \pi_1^s = \frac{(a - c)^2}{8b} > \frac{(a - c)^2}{9b} = \pi_1^c \]
There is first mover advantage

\[ \pi_2^s = \frac{(a - c)^2}{16b} < \frac{(a - c)^2}{9b} = \pi_2^c \]

Example: Ultimatum Bargaining

- Two players, A and B, bargain over a cake of size 1
- Player A makes an offer \( x \in [0, 1] \) to player B
- If player B accepts the offer \( (Y) \), agreement is reached
  - A receives \( x \)
  - B receives \( 1 - x \)
- If player B rejects the offer \( (N) \), both receive zero

Cournot vs. Stackelberg

Backward Induction Equilibrium of Ultimatum Bargaining

- B’s optimal action
  - \( x < 1 \) → accept
  - \( x = 1 \) → accept or reject
- Suppose in equilibrium B accepts any offer \( x \in [0, 1] \)
  - What is the optimal offer by A? \( x = 1 \)
  - The following is a BIE
    \[ x^* = 1 \]
    \[ s_B^*(x) = Y \text{ for all } x \in [0, 1] \]
- Now suppose that B accepts if and only if \( x < 1 \)
  - What is A’s optimal offer?
    - \( x = 1 ? \)
    - \( x < 1 ? \)

Unique BIE

\[ x^* = 1, s_B^*(x) = Y \text{ for all } x \in [0, 1] \]