Games with Incomplete Information

- Some players have incomplete information about some components of the game
  - Firm does not know rival’s cost
  - Bidder does not know valuations of other bidders in an auction
- We could also say some players have private information

Does incomplete information matter?

- Suppose you make an offer to buy out a company
- If the value of the company is \( V \) it is worth \( 1.5V \) to you
- The seller accepts only if the offer is at least \( V \)
- If you know \( V \) what do you offer?
- You know only that \( V \) is uniformly distributed over \([0, 100]\). What should you offer?

Bayesian Games

- We will first look at incomplete information games where players move simultaneously
  - Bayesian games
- Later on we will study dynamic games of incomplete information

What is new in a Bayesian game?

- Each player has a type: summarizes a player’s private information
  - Type set for player \( i \): \( \Theta_i \)
    - A generic type: \( \theta_i \)
  - Set of type profiles: \( \Theta = \times_{i \in N} \Theta_i \)
    - A generic type profile: \( \theta = \{\theta_1, \theta_2, \ldots, \theta_n\} \)
- Players’ payoffs depend on types
  - \( u_i : S \times \Theta \rightarrow \mathbb{R} \)
  - \( u_i(s, \theta) \)
- Incomplete information can be anything about the game
  - Payoff functions
  - Actions available to others
  - Beliefs of others; beliefs of others’ beliefs of others’...
- Harsanyi (1967-1968) showed that introducing types in payoffs is adequate
Bayesian Games

What is new in a Bayesian game?
- Each player has beliefs about others’ types
  - \( p_i : \Theta_i \rightarrow \Delta(\Theta_{-i}) \)
  - \( p_i(\theta_{-i}|\theta_i) \)
- Beliefs are consistent
  - There exists a probability distribution (a common prior) \( p \in \Delta(\Theta) \) such that
    \[
    p_i(\theta_{-i}|\theta_i) = \frac{p(\theta)}{\sum_{t_{-i}} p(\theta_{-i}, t_{-i})}
    \]
    for all \( \theta \in \Theta \) and for all \( i \in N \).

Definition

A **Bayesian Game** is given by

\[
G_B = (N, (S_i), (\Theta_i), (p_i), (u_i))
\]

We say that a Bayesian game is finite if \( N, S_i, \Theta_i \) are finite for all \( i \in N \).

Bayesian Equilibrium

- Since payoffs depend on types different types of the same player may play different strategies
  - \( s_i : \Theta_i \rightarrow S_i \)
  - \( \sigma_i : \Theta_i \rightarrow \Delta(S_i) \)
- We assume that players maximize their expected payoffs. Let
  \[
  U_i(\sigma, \theta) = \sum_{s \in S} \left( \prod_{j \in N} \sigma_j(s_j|\theta_j) \right) u_i(s, \theta)
  \]
  or
  \[
  U_i(\sigma_i, \sigma_{-i}, \theta_i, \theta_{-i}) = \sum_{s \in S} \left( \prod_{j \in N \setminus \{i\}} \sigma_j(s_j|\theta_j) \right) \sigma_i(s_i|\theta_i) u_i(s, \theta_i, \theta_{-i})
  \]

Expected payoff of player \( i \) of type \( \theta_i \) is given by

\[
U_i(\sigma_i, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) U_i(\sigma_i, \theta_i, \theta_{-i})
\]

or

\[
U_i(\sigma_i, \sigma_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) U_i(\sigma_i, \sigma_{-i}, \theta_i, \theta_{-i})
\]

Bank Runs

- You (player 1) and another investor (player 2) have a deposit of $100 each in a bank
- If the bank manager is a good investor you will each get $150 at the end of the year. If not you loose your money
- You can try to withdraw your money now but the bank has only $100 cash
  - If only one tries to withdraw she gets $100
  - If both try to withdraw they each can get $50
- You believe that the manager is good with probability \( q \)
- Player 2 knows whether the manager is good or bad
- You and player 2 simultaneously decide whether to withdraw or not
Bayesian Games

The payoffs can be summarized as follows

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<tbody>
<tr>
<td>Good</td>
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<tr>
<td>Bad</td>
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Two Possible Types of Bayesian Equilibria

- **Separating Equilibria**: Each type plays a different strategy
- **Pooling Equilibria**: Each type plays the same strategy

How would you play if you were Player 2 who knew the banker was bad?

Player 2 always withdraws in bad state: \( \sigma_2(N|B) = 1 \) in any Bayesian equilibrium

Bank Runs

Pooling Equilibria

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Separating Equilibria

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Mixed Strategy Equilibria?

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Is there an equilibrium in which \( \sigma_2(W|G) \in (0,1) \)?

This requires the expected payoffs to \( W \) and \( N \) to be the same for player 2 of Good type:

\[
\sigma_1(W) \times 50 + (1 - \sigma_1(W)) \times 100 = (1 - \sigma_1(W)) \times 150
\]

or \( \sigma_1(W) = 1/2 \)

This, in turn, implies that the expected payoffs to \( W \) and \( N \) must be the same for player 1:

\[
q \times [\sigma_2(W|G) \times 50 + (1 - \sigma_2(W|G)) \times 100] + (1 - q) \times 50 = q \times [\sigma_2(W|G) \times 0 + (1 - \sigma_2(W|G)) \times 150] + (1 - q) \times 0
\]
Mixed Strategy Equilibrium

- This is solved as
  \[ \sigma_2(W|G) = 1 - \frac{50}{100q} \]
- We have
  \[ \sigma_2(W|G) \in (0, 1) \Leftrightarrow q > \frac{1}{2} \]

The following is a Bayesian equilibrium for any \( q > \frac{1}{2} \):\[ \sigma_1(W) = \frac{1}{2}, \sigma_2(W|G) = 1 - \frac{50}{100q}, \sigma_2(W|B) = 1 \]

If \( q < \frac{1}{2} \) the only equilibrium is a bank run.

Cournot Duopoly with Incomplete Information about Costs

- Two firms. They choose how much to produce \( q_i \in \mathbb{R}^+ \)
- Firm 1 has high cost: \( c_H \)
- Firm 2 has either low or high cost: \( c_L \) or \( c_H \)
- Firm 1 believes that Firm 2 has low cost with probability \( \mu \in [0, 1] \)
- payoff function of player \( i \) with cost \( c_j \)
  \[ u_i(q_1, q_2, c_j) = (a - (q_1 + q_2))q_i - c_jq_i \]
- Strategies:
  \[ q_1 \in \mathbb{R}^+, \quad q_2 : \{c_L, c_H\} \rightarrow \mathbb{R}^+ \]

Complete Information

- Firm 1
  \[ \max_{q_1} (a - (q_1 + q_2)) q_1 - c_H q_1 \]
- Best response correspondence
  \[ B_1(q_2) = \frac{a - q_2 - c_H}{2} \]
- Firm 2
  \[ \max_{q_2} (a - (q_1 + q_2)) q_2 - c_j q_2 \]
- Best response correspondences
  \[ B_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}, \quad B_2(q_1, c_H) = \frac{a - q_1 - c_H}{2} \]

Nash Equilibrium

- If Firm 2’s cost is \( c_H \)
  \[ q_1 = q_2 = \frac{a - c_H}{3} \]
- If Firm 2’s cost is \( c_L \)
  \[ q_1 = \frac{a - c_H - (c_H - c_L)}{3}, \quad q_2 = \frac{a - c_H + (c_H - c_L)}{3} \]
Incomplete Information

- Firm 2
  \[
  \max_{q_2} (a - (q_1 + q_2))q_2 - c_jq_2
  \]
- Best response correspondences
  \[
  B_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}
  \]
- Firm 1 maximizes
  \[
  \mu \{ [a - (q_1 + q_2(c_L))]q_1 - c_Hq_1 \}
  + (1 - \mu) \{ [a - (q_1 + q_2(c_H))]q_1 - c_Hq_1 \}
  \]
- Best response correspondence
  \[
  B_1(q_2(c_L), q_2(c_H)) = \frac{a - \mu q_2(c_L) + (1 - \mu)q_2(c_H) - c_H}{2}
  \]

Bayesian Equilibrium

- \[
  q_1 = \frac{a - c_H - \mu(c_H - c_L)}{3}
  \]
- \[
  q_2(c_L) = \frac{a - c_L + (c_H - c_L)}{3} - (1 - \mu)\frac{c_H - c_L}{6}
  \]
- \[
  q_2(c_H) = \frac{a - c_H}{3} + \mu\frac{c_H - c_L}{6}
  \]

- Is information good or bad for Firm 1?
- Does Firm 2 want Firm 1 to know its costs?

Complete vs. Incomplete Information

Purification of Mixed Strategy Equilibria

Interpreting equilibria in mixed strategies is difficult
- individuals flipping coins to determine their actions is counter-intuitive
- why should individuals choose the exact equilibrium probabilities when they are indifferent between actions?

Two responses:
- Interpret the mixed strategy of player \( i \) as the (common) conjecture of other players about player \( i \)'s play

Mixed strategy equilibrium as an equilibrium in conjectures
- Harsanyi's (1973) purification idea

Mixed strategy equilibria as the limit of Bayesian equilibria of perturbed games in which each player is almost always choosing his uniquely optimal action.
Purification of Mixed Strategy Equilibria

- Slightly perturb the payoffs
- Calculate the pure strategy Bayesian equilibria of the games parametrized
  on the perturbations
- Limit of the equilibria as perturbations go to zero is the mixed strategy
  equilibria of the original game
  - A single sequence of perturbed games can be used to purify all the mixed
    strategy equilibria
- We will demonstrate the idea using an example

Purification of Mixed Strategy Equilibria

Consider the Battle of the Sexes game

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<tbody>
<tr>
<td>B</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Three Nash equilibria: \((\sigma_1(B), \sigma_2(B)) \in \{(1,1), (0,0), (2/3, 1/3)\}\)

Perturbed game:

<table>
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<tr>
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<th>B</th>
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<tbody>
<tr>
<td>B</td>
<td>(2 + \varepsilon \theta_1, 1)</td>
<td>(\varepsilon \theta_1, \varepsilon \theta_2)</td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2 + \varepsilon \theta_2</td>
</tr>
</tbody>
</table>

where \(\varepsilon \in (0,1)\) and \(\theta_1\) and \(\theta_2\) are two independently distributed random
variables with uniform distribution over \([0,1]\). This defines a Bayesian game.
We will next find its pure strategy Bayesian equilibria.

Claim

There are three pure strategy Bayesian equilibria of the perturbed Battle of the
Sexes Game:

- \(\sigma_1(B|\theta_1) = \sigma_2(B|\theta_2) = 1\), for all \(\theta_1, \theta_2 \in [0,1]\)
- \(\sigma_1(B|\theta_1) = \sigma_2(B|\theta_2) = 0\), for all \(\theta_1, \theta_2 \in [0,1]\)
- \(\sigma_1(B|\theta_1) = \begin{cases} 1, & \theta_1 > \frac{1}{4+\varepsilon} \\ 0, & \theta_1 \leq \frac{1}{4+\varepsilon} \end{cases}
\sigma_2(B|\theta_2) = \begin{cases} 1, & \theta_2 < \frac{1}{4+\varepsilon} \\ 0, & \theta_2 \geq \frac{1}{4+\varepsilon} \end{cases}\)

Proof

Suppose first that \(\sigma_1(B|\theta_1) = 1\), for all \(\theta_1 \in [0,1]\). Then, it must be that
\(\sigma_2(B|\theta_2) = 1\), for all \(\theta_2 \in [0,1]\). This is indeed an equilibrium.
Second, if \(\sigma_1(B|\theta_1) = 0\), for all \(\theta_1 \in [0,1]\). Then, it must be that \(\sigma_2(B|\theta_2) = 0\), for all
\(\theta_2 \in [0,1]\). This is also an equilibrium.

Proof (cont.)

Last, suppose \(\sigma_1(B|\theta_1) = 1\) for some \(\theta_1\) and \(\sigma_1(B|\theta_1) = 0\) for some other \(\theta_1\).
The payoff matrix makes it clear that then the equilibrium strategy must be
such that there exists a \(\theta_1^* \in (0,1)\) such that

\[\sigma_1(B|\theta_1) = \begin{cases} 1, & \theta_1 > \theta_1^* \\ 0, & \theta_1 \leq \theta_1^* \end{cases}\]

We then have

\[U_2(\sigma_1, B, \theta_2) = 1 \times \int_{\theta_1^*}^{1} d\theta_1 = (1 - \theta_1^*)\]
\[U_2(\sigma_1, S, \theta_2) = \varepsilon \theta_2 \int_{0}^{\theta_1^*} d\theta_1 + (2 + \varepsilon \theta_2) \int_{\theta_1^*}^{1} d\theta_1 = \varepsilon \theta_2 + 2 \theta_1^*\]

Define \(\theta_2^*\) by \(U_2(\sigma_1, B, \theta_2^*) = U_2(\sigma_1, S, \theta_2^*), i.e.,\)

\[\theta_2^* = \frac{1 - 3 \theta_1^*}{\varepsilon}\]
Proof (cont.)

Therefore, we must have

\[ \sigma_2(B|\theta_2) = \begin{cases} 1, & \theta_2 < \theta_2^* \\ 0, & \theta_2 > \theta_2^* \end{cases} \]

We now have

\[ U_1(B, \sigma_2, \theta_1) = (2 + \epsilon \theta_1) \int_0^{\theta_1} d\theta_2 + \epsilon \theta_1 \int_{\theta_2^*}^1 d\theta_2 = \epsilon \theta_1 + 2\theta_2^* \]

\[ U_1(S, \sigma_2, \theta_1) = 1 \times \int_{\theta_2^*}^1 d\theta_2 = (1 - \theta_2^*) \]

We must have \( U_1(B, \sigma_2, \theta_1^*) = U_1(S, \sigma_2, \theta_1^*) \), or

\[ \theta_1^* = \frac{1 - 3\theta_2^*}{\epsilon} \]

Solving for \( \theta_1^* \) and \( \theta_2^* \) we get \( \theta_1^* = \theta_2^* = \frac{1}{3 + \epsilon} \) and this completes the proof. □

As \( \epsilon \to 0 \), we have the following three pure strategy Bayesian equilibria:

\[ \sigma_1(B|\theta_1) = \sigma_2(B|\theta_2) = 1, \text{ for all } \theta_1, \theta_2 \in [0, 1] \]

\[ \sigma_1(B|\theta_1) = \sigma_2(B|\theta_2) = 0, \text{ for all } \theta_1, \theta_2 \in [0, 1] \]

\[ \sigma_1(B|\theta_1) = \begin{cases} 1, & \theta_1 > \frac{1}{3} \\ 0, & \theta_1 \leq \frac{1}{3} \end{cases} \]

\[ \sigma_2(B|\theta_2) = \begin{cases} 1, & \theta_2 < \frac{1}{3} \\ 0, & \theta_2 \geq \frac{1}{3} \end{cases} \]

Therefore, any equilibrium of the BoS game can be obtained as a limit of a pure-strategy equilibrium in the perturbed game as the perturbation goes to zero.

Harsanyi (1973)

“Accordingly, mixed-strategy equilibrium points are stable even though the players may make no deliberate effort to use their pure strategies with the probability weights prescribed by their mixed equilibrium strategies – because the random fluctuations in their payoffs will make them use their pure strategies approximately with the prescribed probabilities.”