Introduction

- Nash equilibrium concept assumes that each player knows the other players’ equilibrium behavior.
- This is problematic at least in one-shot games and when there are multiple equilibria.
- What happens if we relax and use only rationality and common knowledge of rationality?
  - Dominant Strategy Equilibrium
  - Iterated Elimination of Dominated Strategies
  - Rationalizability

Dominant Strategies

Definition (Dominant Strategies)
Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game. A pure strategy $s_i \in S_i$ strictly dominates another pure strategy $s'_i \in S_i$ if

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

$s_i \in S_i$ weakly dominates $s'_i \in S_i$ if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

and

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}) \quad \text{for some } s_{-i} \in S_{-i}$$

A pure strategy $s_i \in S_i$ is strictly dominant if it strictly dominates every $s'_i \in S_i$. It is called weakly dominant if it weakly dominates every $s'_i \in S_i$. 

Consider Prisoners’ Dilemma

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$N$</th>
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</thead>
<tbody>
<tr>
<td>$C$</td>
<td>$-5$</td>
<td>$0$</td>
</tr>
<tr>
<td>$N$</td>
<td>$-6$</td>
<td>$-1$</td>
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</table>

- $C$ is optimal irrespective of what the other player does.
- We call such a strategy a dominant strategy.
- If every player has a dominant strategy, we call the corresponding strategy profile dominant strategy equilibrium.
- It requires only the assumption that players are rational.
- But it usually does not exist.
Dominant Strategy Equilibrium

Definition
A pure strategy profile \( s^* \in S \) is a strictly (weakly) dominant strategy equilibrium of \( G = (N, (S_i), (u_i)) \) if \( s_i^* \) is a strictly (weakly) dominant strategy for each player \( i \in N \). The set of strictly (weakly) dominant strategy equilibria is denoted \( D_s(G) (D_w(G)) \).

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>L</th>
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<tbody>
<tr>
<td>H</td>
<td>10,10</td>
<td>2,15</td>
</tr>
<tr>
<td>L</td>
<td>15,2</td>
<td>5,5</td>
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</tbody>
</table>

- \( L \) strictly dominates \( H \)
- \( D_s(G) = \{(L,L)\} \)

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</tbody>
</table>

- \( L \) weakly dominates \( H \)
- \( D_w(G) = \{(L,L)\} \)

Proposition
Let \( G \) be a strategic form game. \( D_s(G) \subseteq D_w(G) \subseteq N(G) \)

Proof.

Exercise

- A reasonable solution concept
- It only demands the players to be rational
- It does not require them to know that the others are rational
- But it does not exist in many interesting games, e.g., Battle of the Sexes

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
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<tbody>
<tr>
<td>B</td>
<td>2.1</td>
<td>0.0</td>
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<tr>
<td>S</td>
<td>0.0</td>
<td>1.2</td>
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\( D_w(BoS) = \emptyset \)

Consider the following game

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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<tbody>
<tr>
<td>T</td>
<td>3.0</td>
<td>2.1</td>
</tr>
<tr>
<td>M</td>
<td>0.0</td>
<td>3.1</td>
</tr>
<tr>
<td>B</td>
<td>1.1</td>
<td>1.0</td>
</tr>
</tbody>
</table>

- \( B \) is never optimal: \( T \) always does better
  - \( B \) is strictly dominated
    - If player 1 is rational, then she should never play \( B \)
  - If player 2 knows that player 1 is rational, then he knows that she will not play \( B \)
  - Therefore, if player 2 is rational he should never play \( L \)
  - If player 1 knows that
    - player 2 is rational
    - player 2 knows that player 1 is rational
      - then she knows that player 2 will not play \( L \)
      - and therefore should never play \( T \)
  - The unique outcome that survives this elimination process is \((M,R)\)
  - This process is known as Iterated Elimination of Dominated Strategies
Iterated Elimination of Dominated Strategies

- In order to reach a unique outcome we used the following assumptions:
  1. Everybody is rational
  2. Everybody knows that everybody is rational
  3. Player 1 knows that everybody knows that everybody is rational
- In more complicated games it may take more than this
- In the limit we have common knowledge of rationality
  - everybody is rational, everybody knows that everybody is rational,
  - everybody knows that everybody knows that everybody is rational, and so ad infinitum.

Consider the following game

<table>
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<tr>
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</tr>
<tr>
<td>M</td>
<td>0.0</td>
<td>3.1</td>
</tr>
<tr>
<td>B</td>
<td>1.1</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Is there a dominated strategy for any of the players?

- B is strictly dominated by \( \sigma_1 = (1/2, 1/2, 0) \)
- It is possible that a strategy is not strictly dominated by any pure strategy, yet it is strictly dominated
- Let us introduce these concepts a little more formally

Dominated Strategies

**Definition**

Let \( G = (N, (S_i), (u_i)) \) be a strategic form game. A pure strategy \( s_i \in S_i \) is **strictly dominated** if there is a mixed strategy \( \sigma_i \in \Sigma_i \) such that

\[
U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.
\]

\( s_i \) is **weakly dominated** if there is a mixed strategy \( \sigma_i \in \Sigma_i \) such that

\[
U_i(\sigma_i, s_{-i}) \geq U_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.
\]

while

\[
U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i}) \quad \text{for some } s_{-i} \in S_{-i}.
\]

Common knowledge of rationality justifies eliminating dominated strategies iteratively

- This procedure is known as **Iterated Elimination of Dominated Strategies**
- If every strategy eliminated is a strictly dominated strategy
  - **Iterated Elimination of Strictly Dominated Strategies**
- If at least one strategy eliminated is a weakly dominated strategy
  - **Iterated Elimination of Weakly Dominated Strategies**

Guessing Game

- Pick a number between 1 and 99
- You win if your number is closest to 2/3 of the average
- What do you pick?
Never Best Response

- You might object to using mixed strategies to eliminate strategies
- Let us instead agree to eliminate only those strategies that are never optimal regardless of the belief that the player might hold about the play of the others
  - Let us allow a player to believe that others might be coordinating their strategy choices
- We call such strategies never best response

Definition
Let \( G = (N, (S_i), (u_i)) \) be a strategic form game. A pure strategy \( s_i \in S_i \) is a never best response if there is no \( \mu_{-i} \in \Delta(S_{-i}) \) such that
\[
U_i(s_i, \mu_{-i}) \geq U_i(s'_i, \mu_{-i}) \quad \text{for all} \quad s'_i \in S_i.
\]

It turns out that eliminating strictly dominated strategies is equivalent to eliminating never best responses.

Consider again

\[
\begin{array}{ccc}
T & L & R \\
M & 3.0 & 0.1 \\
B & 0.0 & 3.1 \\
1.1 & 1.0
\end{array}
\]

Let \( X \subset \mathbb{R}^n \) be non-empty convex sets with disjoint interiors. Then there exists a hyperplane that separates them, i.e., there exists \( \mu \in \mathbb{R}^n \) such that \( \mu \neq 0 \) and
\[
\mu \cdot x \geq \mu \cdot y, \quad \forall x \in X, \forall y \in Y
\]

Proof
Suppose that \( s'_i \) is strictly dominated by \( \sigma_i \). Then, \( U_i(\sigma, s_{-i}) > U_i(s'_i, s_{-i}), \forall s_{-i}, \) i.e.,
\[
\sum_{s_{-i}} \sigma_i(s_i) U_i(s_i, s_{-i}) > U_i(s'_i, s_{-i}), \forall s_{-i}
\]
This, in turn, implies that for any \( \mu_{-i} \in \Delta(S_{-i}) \)
\[
\sum_{s_{-i}} \mu_{-i}(s_{-i}) \sum_{s_i} \sigma_i(s_i) U_i(s_i, s_{-i}) > \sum_{s_{-i}} \mu_{-i}(s_{-i}) U_i(s'_i, s_{-i})
\]
or
\[
\sum_{s_i} \sigma_i(s_i) U_i(s_i, \mu_{-i}) > U_i(s'_i, \mu_{-i})
\]
Therefore, for any \( \mu_{-i} \in \Delta(S_{-i}) \) there is an \( s_i \in \text{supp}(\sigma_i) \) such that
\[
U_i(s_i, \mu_{-i}) > U_i(s'_i, \mu_{-i})
\]
i.e., \( s'_i \) is a never best response.
Iterated Elimination of Strictly Dominated Strategies

Definition
Let \( G = (N, (S_i), (u_i)) \) be a strategic form game. Let \( X_i(0) = S_i \) and define \( X_i(t) \) as the set of pure strategies that are not strictly dominated in \( X_i(t-1) \), i.e.,

\[
X_i(t) = \{ s_i \in X_i(t-1) : \not\exists \sigma_i \in \Delta(S_i) \text{ s.t. } U_i(\sigma_i, s_{-i}) > U_i(s_i, s_{-i}) \},
\]

for all \( t = 1, 2, \ldots \). Let \( X = \bigcap_{t=0}^{\infty} X_i(t) \). The set of outcomes that survives iterated elimination of strictly dominated strategies in game \( G \) is

\[
X(G) = \times_{i \in N} X_i(t)
\]

Previous proposition implies that we could define \( X_i(t) \) as

\[
X_i(t) = \{ s_i \in X_i(t-1) : \exists \mu_{-i} \in \Delta(X_{-i}(t-1)) \text{ s.t. } U_i(s_i, \mu_{-i}) \geq U_i(s_i', \mu_{-i}), \forall s_i' \in X_i(t-1) \}
\]
Rationalizability

It has been independently developed by
- Bernheim (1984, Econometrica)
- Pearce (1984, Econometrica)

It is based on three basic premises:
- Agents view their opponents’ choices as uncertain events
- Agents are rational
  - they act optimally given their beliefs about these events
- Rationality and preferences are common knowledge

Suppose there are two players: A and B
Rationality of player A implies that she should not play a strategy which is not a best response to some belief she might have about B’s behavior
Furthermore, when forming her beliefs, A should be consistent with B’s rationality, i.e., her beliefs should put zero probability on actions of B which are not best response (for B) to some beliefs held by B regarding A’s behavior
... and so on

Rationalizability

Note that we do not allow players to believe that other players’ strategies maybe correlated
- This is what distinguishes rationalizability and iterated elimination of strictly dominated strategies

Some allow correlation and call it correlated rationalizability
- This is equivalent to iterated elimination of strictly dominated strategies

R is non-empty if $S_i$ is compact for all $i$

Rationalizability

Definition
Let $G = (N, (S_i), (u_i))$ be a strategic form game. Let $R_i(0) = S_i$ and define

$$ R_i(t) = \{ s_i \in R_i(t-1) : \exists \mu_{-i} \in \times_{j \neq i} \Delta(R_j(t-1)), \text{ such that } u_i(s_i, \mu_{-i}) \geq u_i(s'_i, \mu_{-i}) \text{ for all } s'_i \in R_i(t-1) \} $$

for $t = 1, 2, \ldots$. The set of rationalizable strategies for player $i$ is

$$ R_i = \bigcap_{t=0}^{\infty} R_i(t). $$

and the set of rationalizable outcomes of $G$ is

$$ R(G) = \times_{i \in N} R_i. $$

Rationalizability, Nash Equilibrium, and IESDS

Proposition
Let $G$ be a finite strategic form game and $\sigma^* \in N(G)$. Then, $\text{supp}(\sigma^*) \subseteq R(G)$, where $\text{supp}(\sigma^*) = \times_{i \in N} \text{supp}(\sigma^*_i)$.

Proof.
Exercise

Proposition
For any finite strategic form game $G$, if an outcome is rationalizable, then it survives IESDS: $R(G) \subseteq X(G)$.

Proof.
Exercise

Proposition
For any finite two-player strategic form game $G$, $R(G) = X(G)$.
Iterated Elimination of Weakly Dominated Strategies

- Defined in a manner similar to IESDS
- Order of elimination does not matter in IESDS
- It matters in IEWDS

\[
\begin{array}{ccc}
 & L & R \\
U & 3,1 & 2,0 \\
M & 4,0 & 1,1 \\
D & 4,4 & 2,4 \\
\end{array}
\]

- Start with \( U \)
- Start with \( M \)

IEWDS and Nash Equilibrium

**Proposition**

Let \( G \) be a finite strategic form game. If IEWDS results in a unique outcome, then this outcome must be a Nash equilibrium of \( G \).

**Proof.**

Exercise

- Is every outcome that survives IEWDS a Nash equilibrium? No.
- Could a Nash equilibrium involve a weakly dominated strategy? Yes.