Strategic Form Games

- It is used to model situations in which players choose strategies without knowing the strategy choices of the other players
- Also known as normal form games

**Definition (strategic form game)**

A **strategic form game** is composed of:

- Set of players: \( N = \{1, 2, \ldots, n\} \)
- A set of strategies: \( S_i \) for each player \( i \)
- A payoff function: \( u_i : S \rightarrow \mathbb{R} \) for each player \( i \)

It is denoted by \( G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \)

An outcome (or a strategy profile) \( s = (s_1, \ldots, s_n) \) is a collection of strategies, one for each player.

Outcome space

\[ S = \times_{i \in N} S_i = \{ (s_1, \ldots, s_n) : s_i \in S_i, i = 1, \ldots, n \} \]

**Remarks**

- We will sometimes write \( G = (N, (S_i), (u_i)) \), or simply \( G \)
- Payoff functions represent preferences over the set of outcomes and are ordinal (for now)
- If \( S_i \) is finite for all \( i \in N \), then the game is called a **finite game**
- The game is **common knowledge**
- Finite strategic form games with two players can be represented by a bimatrix

**Prisoners’ Dilemma**

- \( N = \{1, 2\} \)
- \( S_i = \{C, N\}, i = 1, 2 \)

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<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
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<tbody>
<tr>
<td></td>
<td>C</td>
<td>N</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>-5, -5</td>
<td>0, -6</td>
</tr>
<tr>
<td><strong>N</strong></td>
<td>-6, 0</td>
<td>-1, -1</td>
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</tbody>
</table>

The following also represents the same game whenever \( a < b < c < d \).
Other Examples

Hawk-Dove

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<thead>
<tr>
<th>Player 1</th>
<th>D</th>
<th>H</th>
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<tbody>
<tr>
<td>D</td>
<td>3.3, 1.5</td>
<td>5.1, 0.0</td>
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Stag Hunt

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<tr>
<th>Player 1</th>
<th>S</th>
<th>H</th>
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<tbody>
<tr>
<td>S</td>
<td>2.2, 0.1</td>
<td>1.0, 1.1</td>
</tr>
</tbody>
</table>

Guessing Game

- There are $n$ players: $N = \{1, 2, \ldots, n\}$
- Each player picks a number between 1 and 99:
  $$S_i = \{1, 2, \ldots, 99\} \text{ for all } i \in N$$
- A player wins if her number is (among the) closest to $2/3$ of the average
  $$u_i(s_1, \ldots, s_n) = \begin{cases} 1, & |s_i - \frac{2}{3} \bar{s}| \leq |s_j - \frac{2}{3} \bar{s}| \text{ for all } j = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$
  where
  $$\bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i$$

Matching Pennies

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<tr>
<th>Player 1</th>
<th>H</th>
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<tbody>
<tr>
<td>H</td>
<td>1, 1</td>
<td>1, -1</td>
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</tbody>
</table>

Levent Kocşesen (Kocş University)

Nash Equilibrium

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Other Examples

Battle of the Sexes

<table>
<thead>
<tr>
<th>Player 2</th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2, 1</td>
<td>0, 0</td>
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Matching Pennies

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Levent Kocşesen (Kocş University)

Nash Equilibrium

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Guessing Game

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  where
  $$\bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i$$

Levent Kocşesen (Kocş University)

Nash Equilibrium

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Cournot Duopoly Model

- There are only two firms
- Inverse (market) demand function: $p(q_1 + q_2)$
  - $p : \mathbb{R}_+ \to \mathbb{R}_+$
  - $p' < 0$
- Cost function of firm $i = 1, 2$: $c_i(q_i)$
  - $c_i : \mathbb{R}_+ \to \mathbb{R}_+$
  - $c_i' > 0, c_i'' \geq 0$

The strategic form is given by
- $N = \{1, 2\}$
- $S_i = \mathbb{R}_+$
- $u_i(q_1, q_2) = p(q_1 + q_2)q_i - c_i(q_i)$ for each $(q_1, q_2) \in S$
Nash Equilibrium

- A solution concept for a strategic for game \( G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \) is a strategy profile \( s \in S \)
- A Nash equilibrium is a strategy profile such that given the strategies of all the other players each player’s strategy maximizes her payoff
- Let 
  \[
  s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)
  \]
  \[
  (s'_i, s_{-i}) = (s_1, s_2, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)
  \]
  \[
  S_{-i} = \times_{j \in N \setminus \{i\}} S_j
  \]

Definition (Nash Equilibrium)

A strategy profile \( s^* \in S \) is a Nash equilibrium of \( G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \) if for each player \( i \in N \)

\[
  u_i(s^*_i, s^*_{-i}) \geq u_i(s'_i, s^*_{-i}) \quad \text{for all } s'_i \in S_i
  \]

The set of Nash equilibria is denoted \( \mathcal{N}(G) \)

Prisoners’ Dilemma (PD)

<table>
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<tbody>
<tr>
<td>C</td>
<td>-5, -5</td>
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<td>-6, 0</td>
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- Is \( (N, N) \) a Nash equilibrium?
- How about \( (C, C) \)?
- \( \mathcal{N}(PD) = \{(C, C)\} \)

Best Response Correspondences

- One way to find Nash equilibria is to first find the best response correspondence for each player
  - Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players
  - ... and then find where they “intersect”

Definition (best response correspondence)

The best response correspondence of player \( i \in N \) is given by \( B_i : S_{-i} \rightarrow S_i \)

such that

\[
  B_i(s_{-i}) = \{ s_i \in S_i : u_i(s_i) \geq u_i(s'_i, s_{-i}) \text{ for all } s'_i \in S_i \}
  \]

Proposition

Let \( G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \) be a strategic form game. A strategy profile \( s^* \in S \) is a Nash equilibrium of \( G \) iff for each player \( i \in N \)

\[
  s^*_i \in B_i(s^*_{-i})
  \]
Examples

Hawk-Dove

\[
\begin{array}{c|cc}
D & H \\ \hline
D & 3,3 & 1,5 \\
H & 5,1 & 0,0
\end{array}
\]

- \( B_1(D) = \{H\} \), \( B_1(H) = \{D\} \)
- \( B_2(D) = \{H\} \), \( B_2(H) = \{D\} \)
- \( N(HD) = \{(H,D),(D,H)\} \)

Battle of the Sexes

\[
\begin{array}{c|cc}
B & S \\ \hline
B & 2,1 & 0,0 \\
S & 0,0 & 1,2
\end{array}
\]

- \( B_1(B) = \{B\} \), \( B_1(S) = \{S\} \)
- \( B_2(B) = \{B\} \), \( B_2(S) = \{S\} \)
- \( N(BS) = \{(B,B),(S,S)\} \)

Nash Equilibria of Cournot Duopoly

Let

\[
p(q_1 + q_2) = \begin{cases} 
  a - b(q_1 + q_2), & q_1 + q_2 \leq a/b \\
  0, & q_1 + q_2 > a/b 
\end{cases}
\]

and for each \( i = 1,2 \)

\[
c_i(q_i) = cq_i
\]

where \( a > c \geq 0 \) and \( b > 0 \)

Therefore, payoff function of firm \( i = 1,2 \) is given by

\[
u_i(q_1, q_2) = \begin{cases} 
  (a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq a/b \\
  -cq_i, & q_1 + q_2 > a/b 
\end{cases}
\]

Claim

Best response correspondence of firm \( i \neq j \) is given by

\[
B_i(q_j) = \begin{cases} 
  \frac{a - c - bq_j}{2b}, & q_j < \frac{a-c}{b} \\
  0, & q_j \geq \frac{a-c}{b}
\end{cases}
\]

Proof.

- If \( q_2 \geq \frac{a-c}{b} \), then \( u_1(q_1, q_2) < 0 \) for any \( q_1 > 0 \). Therefore, \( q_1 = 0 \) is the unique payoff maximizer.
- If \( q_2 < \frac{a-c}{b} \), then the best response cannot be \( q_1 = 0 \) (why?). Furthermore, it must be the case that \( q_1 + q_2 \leq \frac{a-c}{b} \), for otherwise \( u_1(q_1, q_2) < 0 \). So, the following first order condition must hold

\[
\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - 2bq_1 - bq_2 = 0
\]

Similarly for firm 2.

Claim

The set of Nash equilibria of the Cournot duopoly game is given by

\[
N(G) = \left\{ \left( \frac{a-c}{3b}, \frac{a-c}{3b} \right) \right\}
\]

Proof.

Suppose \((q_1^*, q_2^*)\) is a Nash equilibrium and \( q_1^* = 0 \). Then, \( q_1^* = (a-c)/2b < (a-c)/b \). But, then \( q_2^* \notin B_i(q_i^*) \), a contradiction. Therefore, we must have \( 0 < q_i^* < (a-c)/b \), for \( i = 1,2 \). The rest follows from the best response correspondences.
Cournot Nash Equilibrium

Existence of Nash Equilibrium

**Definition (correspondence)**

A correspondence \( \Gamma : X \rightrightarrows Y \) is any mapping that associates with each \( x \in A \) a subset \( \Gamma(x) \) of \( Y \).

**Definition**

\( \Gamma \) is **nonempty and convex-valued** if \( \Gamma(x) \) is nonempty and convex for each \( x \in X \).

**Definition**

\( \Gamma : X \rightrightarrows Y \) has a **closed graph** if, for all \( x \in X \),

\[ x^m \to x, \ y^m \in \Gamma(x^m), \ y^m \to y \Rightarrow y \in \Gamma(x) \]

**Theorem (Kakutani’s Fixed Point Theorem)**

Let \( S \subseteq \mathbb{R}^n \) be a compact and convex set. If \( \Gamma : S \rightrightarrows S \) is a nonempty and convex-valued correspondence with a closed graph, then there exists an \( s \in S \) such that \( s \in \Gamma(s) \).

**Proof.**


**Theorem (Nash’s Existence Theorem)**

Let \( G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \) be a strategic form game such that

- \( S_i \) is a nonempty, convex and compact subset of \( \mathbb{R}^{m_i} \),
- \( u_i \) is continuous on \( S_i \) and quasi-concave on \( S_i \), \( i = 1, \ldots, n \)

Then \( N(G) \neq \emptyset \).

**Proof.**

Define the correspondence \( B : S \rightrightarrows S \) by

\[ B(s) \equiv \{ x \in S : x_1 \in B_1(s_1), x_2 \in B_2(s_2), \ldots, x_n \in B_n(s_n) \} \]

We will show that Kakutani’s theorem applies to \( B \).

- \( S \) is compact and convex (since each \( S_i \) is)
- \( B \) is nonempty-valued (Weierstrass’ theorem)
Nash Equilibrium

Proof (continued)

- $B$ is convex-valued: Take any $x, y \in B(s)$. Then, for all $i \in N$, $u_i(x_i, s_{-i}) = u_i(y_i, s_{-i}) \geq u_i(s_i, s_{-i})$, for all $s_i \in S_i$. Quasi-concavity of $u_i$ on $S_i$ implies $\; u_i(\lambda x_i + (1 - \lambda)y_i, s_{-i}) \geq u_i(x_i, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_i \in S_i, \lambda \in [0, 1].$

Therefore, $\lambda x + (1 - \lambda)y \in B(s)$.

Proof (continued)

- $B$ has closed graph: Take any $s^m \rightarrow s, b^m \rightarrow b, b^m \in B(s^m)$

Need to show: $b \in B(s)$.

Intuitively:

$\; u_i(b^m_i, s^m_{-i}) \geq u_i(x_i, s^m_{-i})$ for all $x_i \in S_i$

$(b^m_i, s^m_{-i}) \rightarrow (b_i, s_{-i})$ and $(x_i, s^m_{-i}) \rightarrow (x_i, s_{-i})$ imply $u_i(b_i, s_{-i}) \geq u_i(x_i, s_{-i})$ for all $x_i \in S_i$, by continuity of $u_i$.

Parametric Games

- Assume that $u_i : S \times \Theta \rightarrow \mathbb{R}$

Then the Nash equilibrium set would depend on $\Theta$

- What happens to the equilibrium set as a result of small changes in $\Theta$?

Proposition

Let $\Theta \neq \emptyset$ be a compact set in $\mathbb{R}^k$ and let $S_i \neq \emptyset$ be a convex and compact subset of $\mathbb{R}^M_i$, $i = 1, \ldots, n$. Assume that $u_i : S \times \Theta \rightarrow \mathbb{R}$ is a continuous function which is quasi-concave on $S_i$ and consider the game $G(\theta) \equiv (N, (S_i), (u_i(\cdot, \theta)))$, where $\theta \in \Theta$. Define the correspondence $\Gamma : \Theta \rightarrow S$ by $\Gamma(\theta) \equiv N(G(\theta))$. Then, $\Gamma$ is a nonempty-valued correspondence with a closed graph.

Proof.

Exercise
Parametric Games: Example

- Let $A = [0, 1]$, $\Theta = [-1, 1]$, $u(x, \theta) = 1 + \theta x$
- Nash equilibrium correspondence:
  \[\Gamma[\theta] = \begin{cases} 
  0, & \theta < 0 \\
  [0, 1], & \theta = 0 \\
  1, & \theta > 0 
\end{cases}\]
- As $\theta \to 0$, Nash equilibrium of $G(\theta)$ converges to an equilibrium of $G(0)$
- There may exist other equilibria of $G(0)$

Symmetric Games

**Definition**

A strategic form game $G = (N, (S_i), (u_i))$ is called symmetric if $S_i = S_j$ and $u_i(s) = u_j(s')$ for all $i, j = 1, \ldots, n$ and all $s, s' \in S$ such that $s'$ is obtained from $s$ by exchanging $s_i$ and $s_j$.

**Prisoners’ Dilemma**

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<tr>
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<tbody>
<tr>
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</tr>
<tr>
<td><strong>D</strong></td>
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Symmetric?

**Battle of the Sexes**

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<tr>
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<th>Player 2</th>
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</thead>
<tbody>
<tr>
<td><strong>B</strong></td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
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</table>

Symmetric?

**Strict Nash Equilibrium**

**Definition**

A strategy profile $s^* \in S$ is a strict Nash equilibrium of $G = (N, (S_i), (u_i))$ if for each player $i \in N$

\[u_i(s^*_i, s^*_{-i}) > u_i(s_i, s^*_{-i})\]

for all $s_i \in S_i$ with $s_i \neq s^*_i$

- Strict Nash equilibria are robust
- Not every game has a strict Nash equilibrium
- Not every Nash equilibrium is strict

<table>
<thead>
<tr>
<th></th>
<th>$U$</th>
<th>$L$</th>
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<tbody>
<tr>
<td><strong>L</strong></td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>0, 0</td>
<td>0, 2</td>
</tr>
</tbody>
</table>

$(U, L)$ is a strict Nash equilibrium
$(D, R)$ is not
What happens if $u_1(U, R) = \varepsilon$, for any $\varepsilon > 0$?
Existence Again

Does the Matching Pennies game have a Nash equilibrium?

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>( H )</td>
</tr>
<tr>
<td>( T )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccc}
1 & -1 & -1, 1 \\
-1, 1 & 1 & -1,
\end{array}
\]

How would you play?
You should try to be unpredictable
Choose randomly

Mixed Strategies

For any finite set \( X \) with \( m \) elements, let \( \Delta(X) \) denote the set of all probability distributions over \( X \)

\[
\Delta(X) = \{ \sigma \in \mathbb{R}_+^m : \sum_{k=1}^m \sigma_k = 1 \}\]

Definition (Mixed Strategies)
Let \( G = (N, (S_i), (u_i)) \) be a strategic form game. A mixed strategy \( \sigma_i \) for player \( i \in N \) is a probability distribution over \( S_i \), i.e., \( \sigma_i \in \Delta(S_i) \).

We assume that mixed strategies are independent across players.

Mixed Strategies

For any \( \sigma_i \in \Delta(S_i) \) and \( s_i \in S_i \), let \( \sigma_i(s_i) \) denote the probability assigned to strategy \( s_i \)

**support** of a mixed strategy is \( \text{supp}(\sigma_i) = \{ s_i \in S_i : \sigma_i(s_i) > 0 \} \)

A pure strategy is a degenerate mixed strategy

- This is the set of strategies that are played with positive probability
- It assigns probability one to only one member of \( S_i \), say \( s_i' \)
- We sometimes refer to it as \( s_i' \)

A mixed strategy profile: \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \times_{i \in N} \Delta(S_i) \)

A mixed strategy profile induces a probability distribution \( p \) over the set of outcomes \( S \)

\[
p(s|\sigma) = \prod_{i \in N} \sigma_i(s_i)
\]

Let

\[
\Sigma_i = \Delta(S_i)
\]

\[
\Sigma = \times_{i \in N} \Delta(S_i)
\]

\[
\Sigma_{-i} = \times_{j \in N \setminus \{i\}} \Delta(S_j)
\]

Assume that players’ preferences are defined over lotteries on \( S \)
and that \( u_i \) is a von Neumann-Morgenstern utility function for each \( i \in N \)
so that they can be represented by the expected payoff function

\[
U_i(\sigma) = \sum_{s \in S} p(s|\sigma) u_i(s)
\]

= \[
\sum_{s \in S} \left( \prod_{j \in N} \sigma_j(s_j) \right) u_i(s)
\]

Definition
Let \( G = (N, (S_i), (u_i)) \) be a strategic form game. The mixed extension of \( G \) is given by \( \Gamma(G) = (N, (\Sigma), (U_i)) \).
Mixed Strategies

Claim

\( U_i \) is multilinear in each component of a strategy profile. For any \( j \in N, \sigma^_, \sigma^i \in \Sigma_j, \sigma^j \in \Sigma_i \), and \( \lambda \in [0, 1] \), the following is true:

\[
U_i(\lambda \sigma^i + (1 - \lambda) \sigma^j, \sigma^j) = \lambda U_i(\sigma^j, \sigma^j) + (1 - \lambda) U_i(\sigma^i, \sigma^j)
\]

Claim

Let \( s_i \) be a pure strategy and \( \sigma \) a mixed strategy profile. Then

\[
U_i(\sigma) = \sum_{s_i \in \Sigma_i} \sigma_i(s_i) U_i(s_i, \sigma^j)
\]

Theorem

Every finite strategic form game has a mixed strategy equilibrium.

Proof.

Exercise

- From now on we use Nash equilibrium and mixed strategy equilibrium interchangeably.
- If every player's strategy in a Nash equilibrium is a pure strategy we call it a pure strategy equilibrium.
- Matching pennies game does not have any pure strategy equilibrium.
- but by the above theorem it must have a (mixed strategy) Nash equilibrium.
- The following proposition is very useful in calculating mixed strategy equilibria.

Mixed Strategy Equilibrium

Definition

The set of mixed strategy equilibria of a finite strategic form game \( G \) is the set of Nash equilibria of the mixed extension of \( G \). In other words, \( \sigma^* \in \Sigma \) is a mixed strategy equilibrium of \( G \) if, and only if, for every \( i \in N \)

\[
U_i(\sigma^*) \geq U_i(\sigma_i, \sigma^*_{-i}), \text{ for all } \sigma_i \in \Delta(S_i)
\]

Definition (best response correspondence)

The best response correspondence of player \( i \in N \) is given by \( B_i : \Sigma_{-i} \rightarrow \Sigma_i \) such that

\[
B_i(\sigma_{-i}) = \{ \sigma_i \in \Sigma_i : U_i(\sigma) \geq U_i(\sigma'_i, \sigma_{-i}) \text{ for all } \sigma'_i \in \Sigma_i \}
\]

Proposition

Let \( G \) be a finite strategic form game. A mixed strategy profile \( \sigma^* \in \Sigma \) is a mixed strategy equilibrium of \( G \) if, and only if, for each player \( i \in N \) and for any \( s_i \in \text{supp}(\sigma^*_i) \)

\[
s_i \in B_i(\sigma^*_{-i})
\]

Proof.

\((\Rightarrow)\)

Let \( \sigma^* \) be a mixed strategy equilibrium. Suppose that for some player \( i \) and some \( s_i \in \text{supp}(\sigma^*_i) \) we have \( s_i \notin B_i(\sigma^*_{-i}) \), i.e., there exists \( s'_i \in S_i \) such that

\[
U_i(s'_i, \sigma^*_{-i}) > U_i(s_i, \sigma^*_{-i}).
\]

But then, player \( i \) can increase her payoff by shifting some probability from \( s_i \) to \( s'_i \), a contradiction.

\((\Leftarrow)\)

Suppose now that for each player \( i \in N \) and for any \( s_i \in \text{supp}(\sigma^*_i) \) we have \( s_i \notin B_i(\sigma^*_{-i}) \), but \( \sigma^* \) is not a mixed strategy equilibrium. Then, there is an \( i \in N \) and \( \sigma'_i \in S_i \) such that

\[
U_i(\sigma'_i, \sigma^*_{-i}) > U_i(\sigma^*_i, \sigma^*_{-i}).
\]

But then, there must be a \( s'_i \in \text{supp}(\sigma^*_i) \) and \( s_i \in \text{supp}(\sigma^*_i) \) such that \( U_i(s'_i, \sigma^*_{-i}) > U_i(s_i, \sigma^*_{-i}) \), a contradiction.
Matching Pennies

- Suppose $\sigma^*$ is a Nash equilibrium of this game.
  - It must be that $\text{supp}(\sigma^*_i) = \{H, T\}, i = 1, 2$. Why?
  - By the above proposition $U_1(H, \sigma^*_2) = U_1(T, \sigma^*_2)$ or
    \[
    \sigma^*_2(H) \times 1 + (1 - \sigma^*_2(H)) \times (-1) = \sigma^*_2(H) \times (-1) + (1 - \sigma^*_2(H)) \times 1
    \]
    which implies $\sigma^*_2(H) = 1/2$
  - Similarly, we have $\sigma^*_1(H) = 1/2$. So,
    \[
    N(MP) = \left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right\}
    \]

Mixed and Pure Strategy Equilibria

- How do you find the set of all (pure and mixed) Nash equilibria?
- In $2 \times 2$ games we can plot the best response correspondences and find where they intersect.
- Consider the Battle of the Sexes game

\[
\begin{array}{c|cc}
B & S \\
\hline
B & 2, 1 & 0, 0 \\
S & 0, 0 & 1, 2 \\
\end{array}
\]

- Simplify notation: $p = \sigma_1(B)$ and $q = \sigma_2(B)$

**Player 1’s best response:**

- What is Player 1’s best response?
  - Expected payoff to
    - If $2q > 1 - q$ or $q > 1/3$ then the best response is $B$ (or equivalently $p = 1$)
    - If $2q < 1 - q$ or $q < 1/3$ then the best response is $S$ (or equivalently $p = 0$)
    - If $2q = 1 - q$ then she is indifferent
  - Player 1’s best response correspondence:
    \[
    B_1(q) = \begin{cases} 
    \{1\} & \text{if } q > 1/3 \\
    [0, 1] & \text{if } q = 1/3 \\
    \{0\} & \text{if } q < 1/3 
    \end{cases}
    \]

**Player 2’s best response:**

- What is Player 2’s best response?
  - Expected payoff to
    - If $p > 2(1 - p)$ or $p > 2/3$ then the best response is $B$ (or equivalently $q = 1$)
    - If $p < 2(1 - p)$ or $p < 2/3$ then the best response is $S$ (or equivalently $q = 0$)
    - If $p = 2(1 - p)$ then she is indifferent
  - Player 2’s best response correspondence:
    \[
    B_2(p) = \begin{cases} 
    \{1\} & \text{if } p > 2/3 \\
    [0, 1] & \text{if } p = 2/3 \\
    \{0\} & \text{if } p < 2/3 
    \end{cases}
    \]
\[ B_1(q) = \begin{cases} 
\{1\}, & \text{if } q > \frac{1}{3} \\
[0, 1], & \text{if } q = \frac{1}{3} \\
\{0\}, & \text{if } q < \frac{1}{3} 
\end{cases} \]

\[ B_2(p) = \begin{cases} 
\{1\}, & \text{if } p > \frac{2}{3} \\
[0, 1], & \text{if } p = \frac{2}{3} \\
\{0\}, & \text{if } p < \frac{2}{3} 
\end{cases} \]

Set of Nash equilibria:
\[ \{(0, 0), (1, 1), (2/3, 1/3)\} \]