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REAL-TIME SIMULATION OF DYNAMICALLY DEFORMABLE FINITE ELEMENT MODELS USING MODAL ANALYSIS and SPECTRAL LANCZOS DECOMPOSITION METHODS

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Real-time simulation of deformable objects using finite element models is a challenge in medical simulation. We present two efficient methods for simulating real-time behavior of a *dynamically* deformable 3D object modeled by finite element equations. The first method is based on modal analysis, which utilizes the most significant vibration modes of the object to compute the deformation field in real-time for applied forces. The second method uses the spectral Lanczos decomposition to obtain the explicit solutions of the finite element equations that govern the dynamics of deformations. Both methods rely on modeling approximations, but generate solutions that are computationally faster than the ones obtained through direct numerical integration techniques. In both methods, the errors introduced through approximations were insignificant compare to the computational advantage gained for achieving real-time update rates.

1. Physically-based modeling of deformable objects for medical simulation

Simulation of soft tissue behavior in real-time is a challenging problem. Once the contact between an instrument and tissue is determined, the problem centers on tool-tissue interactions. This involves a realistic haptic feedback to the user and a realistic graphical display of tissue behavior depending on what surgical task (e.g. suturing, grasping, cutting, etc.) the user chooses to perform on the tissue. This is a nontrivial problem which calls for prudence in the application of mechanistic and computer graphics techniques in an endeavor to create a make-believe world that is realistic enough to mimic reality but efficient enough to be executable in real time. Soft-tissue mechanics is complicated not only due to non-linearities, rate and time dependence in material behavior, but also because the tissues are layered and non-homogeneous. The finite element methods (FEM), though they demand more CPU time and memory, seem promising in integrating tissue characteristics into the organ models. Although mechanics community has developed sophisticated tissue models based on FEM, their integration with medical simulators has been difficult due to real-time requirements. Simulating the real-time deformable dynamics of a 3D object using FEM is increasingly more difficult as the total number of nodes/degrees of freedom (dof) increase. With the addition of haptic displays, this has been even more challenging since a haptic loop typically requires a much higher update rate than a visual loop for stable force interactions. Although fast finite element models have been developed for medical applications (Bro-Nielsen and Cotin, 1996; Berkley et al., 2000), less attention has been paid to displaying *time dependent* deformations of large size models in real-time. This paper introduces two numerically fast techniques for real-time simulation of ***dynamically*** deformable (i.e. time-dependent deformations) 3D objects modeled by FEM: (a) Modal analysis (Basdogan, 1999; Basdogan et al., 2000) and (b) spectral Lanczos Decomposition.

2. Our Finite Element Model

The finite element formulation: The 3D models of organs used in our simulations were constructed from discrete triangular surface elements interconnected to each other through nodal points. The coordinates of vertices (nodal points), the polygon indexing, and the connectivity of vertices were derived from the geometric model of each organ. In order to analyze the deformations of the organs under various loading conditions, we considered a combination of membrane and bending elements in our finite element model. This facilitated the continuity in the formulation and enabled us to compute the displacements of nodal points in X, Y, Z directions for both inplane and bending loads. For each node of the triangular element subjected to inplane loads, the displacements in the local x and y directions were taken as the degrees of freedom. The resulting system equations were expressed in local coordinate system as

$$F_m = \left[k^e{}_m\right] U_m \qquad \text{(Eq. 1)}$$

where, m represents the membrane action. Similarly, for each node of the triangular element subjected to bending loads, the displacement in local direction and rotations about the local x and y axes were considered. The relation between the vertex displacements and forces were written as

$$F_b = \left[k^e{}_b\right] U_b \qquad \text{(Eq. 2)}$$

where, b represents the bending action. In order to obtain the local stiffness matrix for each triangular element, the inplane and bending stiffness matrices were combined. Since 6 degrees of freedom were assumed for each of the vertices, the resulting combined local stiffness matrix ($\left[k^e\right]$) became 18x18 for each triangular element.

Transformations: The stiffness matrix derived in the previous section utilizes a system of local coordinates. However, the geometric model of each organ was generated based on the global coordinate system. In order to apply the computations described in the previous section, a transformation from the global coordinates to local coordinate system was required (Zienkiewicz, 1990). It was also necessary to transform the results back to the global reference frame to display the deformations following the solution of finite element equations. The new positions of each nodal point for a given force were computed using system equations in the local coordinate system. Then, these new coordinates were transformed to the global coordinate system in order to update the graphics.

Assembly of Element Stiffness Matrices: The element stiffness matrices (K^e) were put together to construct the overall stiffness matrix (K). This process can be symbolically written as

$$K = \sum_{e=1}^{p} K^e \qquad \text{(Eq. 3)}$$

where, p represents the number of triangles (Rao, 1988).

Implementation of Boundary Conditions: In order to obtain a unique solution for finite element equations, at least one boundary condition must be supplied. The boundary conditions modify the stiffness matrix K and make it nonsingular. There are multiple ways of implementing boundary conditions (Huebner et al., 1995). The easiest way to implement the boundary conditions is to modify the diagonal elements of the K matrix and the rows of the force vector F at which the boundary conditions will be applied. In our model, at least one end of the organ was always fixed, which implied zero displacements for the associated fixed nodes. To implement this boundary condition, diagonal elements of the K matrix and the rows of the F vector associated with those fixed nodes were multiplied by a large number. This procedure makes the unmodified terms of K very small compared to the modified ones.

Eliminating the Rotational Degrees of Freedom (Condensation): The global stiffness matrix was assembled as a symmetric square matrix and its length was six times the number of nodes of the object (recall that we defined 6-dof, 3 translations and 3 rotations, for each node of the object). Although the rotational dof were necessary for the continuity of the solution, their computation was not required for our simulations. Since our main interest was to obtain the translational displacements of each node, the overall stiffness matrix K was condensed such that the rotational dof were eliminated from the formulation. In addition, the condensation of K matrix automatically reduced the number of computations to half, which was helpful for achieving real-time rendering rates. To condense the K matrix, we first partitioned the displacement and load vectors of the static problem as

$$\begin{bmatrix} K_{tt} & K_{tr} \\ K_{rt} & Krr \end{bmatrix} \begin{bmatrix} U_t \\ U_r \end{bmatrix} = \begin{bmatrix} F_t \\ F_r \end{bmatrix} \qquad \text{(Eq. 4)}$$

where, subscripts t and r represent the translational and rotational dof respectively. Then, we set the forces acting on the rotational degrees of freedom to zero, and condensed the stiffness matrix as

$$K_{condensed} = K_{tt} - K_{tr}(K_{rr})^{-1} K_{rt} \qquad \text{(Eq. 5)}$$

The condensed stiffness matrix was a full square matrix and its length was three times the number of nodes of the object.

3. Modal Analysis Method

The dynamic equilibrium equations for a deformable body modeled by FEM can be written as

$$M\ddot{U} + B\dot{U} + KU = F \qquad \text{(Eq. 6)}$$

where, M and B represent the mass and the damping matrices respectively. Once the equations of motion for deformable body are derived, the solution is typically obtained using numerical techniques. The real-time display of FEM becomes increasingly more difficult as the number of elements is increased. However, a particular choice of the mass and damping matrices reduces the number of computations significantly. If the mass matrix is assumed to be diagonal (mass is concentrated at the nodes) and the damping matrix is assumed to be linearly proportional with the mass matrix ($B = \alpha M$), the equations are greatly simplified. A further modeling simplification can be implemented if we assume that high frequency vibration modes contribute very little to the computation of deformations and forces. If dynamic equilibrium equations are transformed into a more effective form, known as modal analysis, fast real-time solutions can be obtained with very reasonable accuracy. Pentland and Williams (1989) demonstrated the implementation of this technique in graphical animation of 3D objects. In modal analysis, global coordinates are transferred to modal coordinates to decouple the differential equations. Then, one can either obtain the explicit solution for each decoupled equation as a function of time or integrate the set of decoupled equations in time to obtain the displacements and forces. Moreover, we can also reduce the dimension of the system, as well as the number of computations, by picking the most significant vibration modes of the object and re-arranging the mass, damping, and stiffness matrices. This procedure is also known as modal reduction.

3.1. Modal Transformation

We defined the following transformation to transform our differential system into a modal system:

$$U(t)_{nx1} = \Phi_{nxn} X(t)_{nx1} \qquad \text{(Eq. 7)}$$

where, Φ is the modal matrix, U and X represent the original and modal coordinates respectively. The modal matrix was obtained by solving the eigen problem for free undamped equilibrium equations:

$$K\phi = \omega^2 M\phi \qquad \text{(Eq. 8)}$$

where, ω and ϕ represent the eigenvalues (i.e. vibration frequencies) and eigenvectors (i.e. mode shapes) of the matrix ($M^{-1}K$) respectively. The modal matrix was constructed by first sorting the frequencies in ascending order and then placing the corresponding eigenvectors into the modal matrix in column-wise format ($0 \le \omega_1 \le \omega_2 \le \omega_3 \ldots \le \omega_n$, $\Phi = [\phi_1, \phi_2, \phi_3, \ldots, \phi_n]$). Finally, a set of decoupled differential equations (i.e. modal system) was obtained using the modal matrix and the transformation defined by Eq. 7:

$$\ddot{X}_i + \alpha_i \dot{X}_i + \omega_i^2 X = f_i \qquad i = 1, \ldots, n \qquad \text{(Eq. 9)}$$

where, n is the degrees of freedom (dof) of the system, $\alpha_i = 2\omega_i\zeta_i$, and $f_i = \phi_i^T F$ are the modal damping and force respectively. Note that ζ is known as the damping ratio or modal damping factor.

3.2. Modal Reduction

Once the equations for modal system are derived, the explicit solutions can be obtained using the Duhamel integral (see Bathe 1996). Alternatively, one can use numerical integration techniques to obtain the modal solution. At this stage, we can also implement the modal reduction approach to significantly reduce the number of computations. For a deformable body under external loading, the high frequency modes do not significantly contribute to the displacements. Hence, the final, deformed shape of the object can be approximated by "r" number of low frequency modes. To implement the modal reduction, we picked the most significant vibration modes of the object (i.e. the first "r" columns of the modal matrix). As a result, our differential system was reduced to "r" number of equations, which were solved using the Newmark numerical integration technique:

$$\ddot{X}^R{}_i + \alpha \dot{X}^R{}_i + \omega_i^2 X^R = f^R{}_i \qquad i = 1, \ldots, r \qquad \text{(Eq. 10)}$$

where, the superscript R represents the reduced system. We then transferred the modal solutions back to the original coordinate frame using the following transformation:

$$U(t)_{nx1} = \Phi^R{}_{nxr} X^R(t)_{rx1} \qquad \text{(Eq. 11)}$$

3.3. Numerical Integration

Numerical integration techniques are typically used to solve the differential equations that arise from finite element models. Various integration schemes based on finite difference techniques have been suggested in the literature for the dynamic analysis of FEM (Bittnar and Sejnoha, 1996; Bathe 1996). In surgical simulation, real-time performance and the stability of solutions for various loading, initial, and boundary conditions are equally important. For example, the central difference method appears to be fast and simple to implement, but the solutions become unstable if the integration step (Δt) is greater than (T_n/π), where T_n is the shortest period of vibration. Bathe (1996) suggests Newmark numerical integration procedure due to its favorable stability and accuracy characteristics. Using the Newmark method, we first formulated the displacement and velocity of each reduced modal coordinate at $t+\Delta t$ as

$$^{t+\Delta t}\dot{X}^{R} = {}^{t}\dot{X}^{R} + [(1-\delta)\,{}^{t}\ddot{X}^{R} + \delta\ {}^{t+\Delta t}\ddot{X}^{R}]\,\Delta t \qquad \text{(Eq. 12)}$$

$$^{t+\Delta t}X^{R} = {}^{t}X^{R} + {}^{t}\dot{X}^{R}\,\Delta t + [(\frac{1}{2}-\eta)\,{}^{t}\ddot{X}^{R} + \eta\ {}^{t+\Delta t}\ddot{X}^{R}]\,\Delta t^{2} \qquad \text{(Eq. 13)}$$

where, η and δ are parameters that can be determined to obtain integration accuracy and stability (solutions become unconditionally stable for $\eta = 1/4$ and $\delta = 1/2$). Then, the equilibrium equation for each reduced modal coordinate was formulated at $t+\Delta t$ as

$$^{t+\Delta t}\ddot{X}^{R} + \alpha\ {}^{t+\Delta t}\dot{X}^{R} + \omega^{2}\ {}^{t+\Delta t}X^{R} = {}^{t+\Delta t}f^{R} \qquad \text{(Eq. 14)}$$

Finally, we substituted the displacement and velocity formulations into the equilibrium equation derived for $t+\Delta t$ and obtained a system that looks quite similar to the static analysis:

$$\hat{F}\hat{U} = \hat{K} \qquad \text{(Eq. 15)}$$

where, $\hat{F}, \hat{U}, \hat{K}$ are modified force and displacement vectors and modified stiffness matrix.

4. The Spectral Lanczos Decomposition Method (SLDM)

Druskin and Knizhnerman (1994) have recently introduced a new technique, called Spectral Lanczos Decomposition method (SLDM), to solve the Maxwell's diffusion equations for multiple frequencies with negligible additional computation. Zunoubi et al. (1998) have demonstrated the efficiency of this technique by studying the resonant frequencies of various microwave cavities. We have followed their approach with some modifications to find the explicit solutions of our finite element equations. In order to solve the finite element equations using the SLDM, we first rearrange the terms of the finite element equations as:

$$\left[\frac{\partial}{\partial t^{2}} I + \alpha \frac{\partial}{\partial t} I + K'\right] E' = F' \qquad \text{(Eq. 16)}$$

where, $K' = M^{-1/2} K M^{-1/2}$, $E' = M^{1/2} U$, $F' = M^{-1/2} F$. If we transfer the equations to Laplace domain and assume that the applied force is constant for a short period of time with a magnitude of F_o, we obtain $E'(s) = F_o/(s(s^2 I + \alpha\, s + K'))$. Using the separation of variables:

$$E'(s) = A/s + (Bs + C)/(s^2 I + \alpha s + K') \qquad \text{(Eq. 17)}$$

where, $A = F_o/K'$, $B = -F_o/K'$ and $C = -(\alpha\, F_o)/K'$. Then, if we apply the inverse Laplace transform, we obtain:

$$E'(t) = F_o \frac{1}{K'}(1 - e^{\frac{-\alpha t}{2}} Cos(\sqrt{K' - (\alpha^2/4)I}\ t) + \frac{\alpha}{2} \frac{1}{\sqrt{K' - (\alpha^2/4)I}} e^{\frac{-\alpha t}{2}} Sin(\sqrt{K' - (\alpha^2/4)I}\ t)$$

(Eq. 18)

Now, if K' matrix is approximated as a diagonal matrix, we can easily obtain the time domain solutions. To achieve our goal, we implement the classical Lanczos scheme (Datta, 1994) with complete reorthogonalization using Householder transformations (Golub, 1996). For this purpose, we first compute the tridiagonal Ritz approximation (T) of the matrix K':

$$Q^T K' Q = T \quad \text{(Eq. 19)}$$

where, $Q = [q_1, q_2, \ldots, q_M]$ is an orthogonal matrix (The vectors $q_1, q_2, \ldots, q_M$ are called Lancsoz vectors), M is the size of the square K'matrix (also the number of equations) and T is the tridiagonal matrix which is determined using the complete reorthogonalization Lanczos scheme that relies on repetitive Householder transformations (Golub, 1996). If we define the Λ and V are the eigen-values and -vectors of the matrix T, one can then write matrix T as:

$$T = V \Lambda V^T \quad \text{(Eq. 20)}$$

where, $\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_M]$. Finally, $E'(t)$ can be approximated as:

$$E'(t) = F_o QV \left[\frac{1}{\Lambda}(1 - e^{\frac{-\alpha t}{2}} Cos(\sqrt{\Lambda - (\alpha^2/4)}\, t) + \frac{\alpha}{2} \frac{1}{\sqrt{\Lambda - (\alpha^2/4)}} e^{\frac{-\alpha t}{2}} Sin(\sqrt{\Lambda - (\alpha^2/4)}\, t) \right] V^T e_1$$

(Eq. 21)

where, $e_1 = (1,0,0,\cdots,0)^T$ is a unit vector.

4.1. Superposition

After obtaining the explicit solutions of the finite element equations, we generate, *an "impedance map"* of the 3D object. This involves the pre-computation of displacement fields (i.e. a look-up table) by applying unit loads along each nodal degrees of freedom, while assuring the positive definiteness of the structure. Such a look-up table can be pre-computed well ahead of the actual interactions. We used this look-up table in conjunction with the "superposition" technique to calculate the deformations of the object for applied loads. The superposition approach calculates the response of the complete system by superimposing (i.e., adding together) the individual responses of the nodes. To calculate the response of a certain node, only the responses of neighboring nodes were used (i.e. define a *radius of influence* and consider the contribution of nodes which are within the radius of influence). The superposition approach provides a solution, which is an approximation to the exact solution, but this approximation is reasonably accurate (please note that we use a linear finite element model for simulating the behavior of tissues and this approximation will not work well with a nonlinear model).

5. Discussion

In this study, a modal analysis and the Spectral Lanczos Decomposition method were proposed to simulate the *dynamic* deformations of a 3D object modeled using finite elements. Although the proposed methods can simulate the real-time dynamics of a deformable object, our finite element model only approximates the characteristics of living tissues with a certain degree due to the stringent requirements of real-time simulation. However, we should point out that both methods are computationally faster than the direct numerical integration methods. For example, we have observed that the direct numerical integration of the original differential system results in O(n^2) floating point operations (flops) where n is the degrees of freedom or the number of nodes of the object. However, the solutions generated using modal analysis lead to O($n \log n$) flops. Therefore, the real-time simulation of finite element models using the direct integration techniques will be increasingly more difficult as n increases. While the proposed modal approach enables to compute real-time solutions numerically, the SLDM can easily return the explicit solutions of the finite element equations for various frequencies. The SLDM, when combined with the superposition technique, is very efficient in simulating real-time deformations of objects. A pre-computed "*impedance* map" of an object using the SLDM enables us to estimate the deformation field of the object easily at the contact point during the real-time interactions.

References:

1. Basdogan C., 2000, Course Name: Simulating Minimally Invasive Surgical Procedures in Virtual Environments: from Tissue Mechanics to Simulation and Training, Medicine Meets Virtual Reality (MMVR'2000), Jan. 27-30, Irvine, CA, http://www.amainc.com/MMVR/MMVR2000pro.html http://eis.jpl.nasa.gov/~basdogan/Tutorials/alltutorials.html (tutorial notes)
1. Basdogan, C., Ho, C., Srinivasan, M.A., 2000, "Virtual Environments for Medical Training: Graphical and Haptic Simulation of Laparoscopic Common Bile Duct Exploration", submitted to IEEE/ASME Transactions on Mechatronics.
2. Datta, B. N., 1994, "Numerical Linear Algebra and Applications", (Chapter 8: Numerical Matrix Eigenvalue Problems).
3. Druskin, V., Knizhnerman, L., 1994, "Spectral Approach to Solving Three-Dimensional Maxwell's Diffusion Equations in the Time and Frequency Domains", Radio Science, Vol. 29, No. 4, pp. 937-953.
4. Zunoubi, M., Donepudi, K.C., Jin, J.M., Chew W.C., 1998, "Efficient Time-Domain and Frequency-Domain Finite-Element Solution of Maxwell's Equations Using Spectral Lanczos Decomposition Method", IEEE Transactions on Microwave Theory and Techniques", Vol. 46, No.8, 1141-1149.
5. Golub, G.H., Van Loan C., 1996, "Matrix Computations", The Johns Hopkins University Press, Baltimore (see Chapters 5.1 and 9.2)
6. Rao, S.S., (1988), "The finite element method in engineering", Pergamon Press, New York.
7. Zienkiewicz, O.C., (1979), "The finite element method", McGraw-Hill, New Delhi.
8. Bathe, K., (1996), "Finite Element Procedures", Prentice Hall, New Jersey.
9. Bittnar, Z., Sejnoha, J., 1996, "Numerical Methods in Structural Mechanics", ASCE Press, New York.
10. Huebner, K.H., Thornton, E.A., Byrom, T.G., 1995, "The finite element method for engineers", John Wiley & Sons, Inc., New York.
11. Pentland, A., Williams, J., 1989, "Good Vibrations: Modal Dyanmics for Graphics and Animation", SIGGRAPH Proceedings, Vol. 23, No. 3, pp.215-222.
12. Bro-Nielsen, M., Cotin, S., 1996, "Real-time Volumetric Deformable Models for Surgery Simulation using Finite Elements and Condensation", EUROGRAPHICS'96, Vol. 15, No. 3, pp. 57-66.
13. Berkley, J., et al., 2000, "Creating Fast Finite Element Models from Medical Images", Proceedings of Medicine Meets Virtual Reality 2000, pp. 26-32.