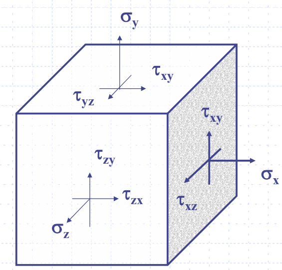
Three-Dimensional Stress Analysis



Three-dimensional stress on an element.

Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$

Strain Displacement

(u,v,w) are the x, y and z components of displacement.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Stress-Strain Relationships

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}$$

3D Stress-Strain Matrix

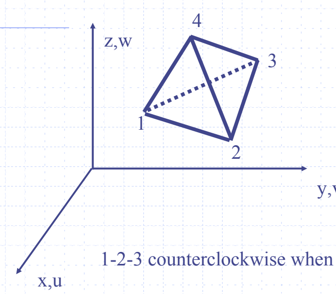
$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}$$

Note: $G = \frac{E}{2(1+\nu)}$

Step 1 - Select the Element Type

1. Discretize Body into Four-Noded Tetrahedral Elements.
2. Three degrees-of-freedom per node.
3. These are x , y, and z displacements.
4. u_i - x displacement at i^{th} node.
5. v_i - y displacement at i^{th} node.
6. w_i - z displacement at i^{th} node.

Tetrahedral Element



1-2-3 counterclockwise when viewed form 4.

Step 2 - Select Displacement Function

Select Linear Functions:

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$$
$$v(x,y,z) = a_5 + a_6 x + a_7 y + a_8 z$$
$$w(x,y,z) = a_9 + a_{10} x + a_{11} y + a_{12} z$$

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$u(x,y,z) = \frac{1}{6V}\begin{Bmatrix} (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)u_1 + (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)u_2 + \\ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)u_3 + (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)u_4 \end{Bmatrix}$$

$$v(x,y,z) = \frac{1}{6V}\begin{Bmatrix} (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)v_1 + (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)v_2 + \\ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)v_3 + (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)v_4 \end{Bmatrix}$$

$$w(x,y,z) = \frac{1}{6V}\begin{Bmatrix} (\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)w_1 + (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)w_2 + \\ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)w_3 + (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)w_4 \end{Bmatrix}$$

V = Volume of tetrahedron.

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

$$\alpha_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$\alpha_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$\alpha_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad \beta_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$$

$$\gamma_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} \qquad \delta_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$N_1 = \frac{(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)}{6V}$$

$$N_2 = \frac{(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)}{6V}$$

$$N_3 = \frac{(\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)}{6V}$$

$$N_4 = \frac{(\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)}{6V}$$

Step 3 - Define Strain/Displacements and Stress/Strain Relationships

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{Bmatrix}$$

$$[B] = [B_1 \;\vdots\; B_2 \;\vdots\; B_3 \;\vdots\; B_4]$$

$$[B_1] = \begin{bmatrix} N_{1,x} & 0 & 0 \\ 0 & N_{1,y} & 0 \\ 0 & 0 & N_{1,z} \\ N_{1,y} & N_{1,x} & 0 \\ 0 & N_{1,z} & N_{1,y} \\ N_{1,z} & 0 & N_{1,x} \end{bmatrix}$$

$$[B_1] = \frac{1}{6V} \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \delta_1 \\ \gamma_1 & \beta_1 & 0 \\ 0 & \delta_1 & \gamma_1 \\ \beta_1 & 0 & \delta_1 \end{bmatrix}$$

Stress/Strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Step 4 - Element Stiffness

$$\iiint_V [B]^T [D] \; [B] \; dV \; \{d\} = \{f\}$$

$$[k] = \iiint_V [B]^T [D] \; [B] \; dV$$

[B] and [D] are constant, therefore:

$$[k] = [B]^T [D] \; [B] \; V$$

Body Forces

$$\{f_b\} = -\iiint_V [N]^T \{X\} \; dV$$

Surface Forces

$$\{f_s\} = -\iint_S [N]^T \{T\} \; dS$$

Uniform pressure on surface 1-2-3

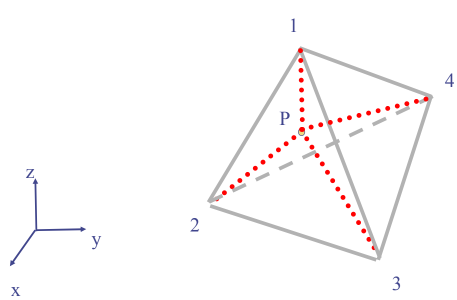
$$\{f_s\} = -\iint_S [N]^T \{T\}\, dS$$

$$\{f_s\} = -\iint_S [N]^T \Big|_{\substack{\text{evaluated on}\\ \text{surface } 1,2,3}} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} dS$$

$$\{f_s\} = \frac{S_{123}}{3} \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

S_{123} = Area of surface 123

Volume Coordinates



At a point P, four tetrahedrons can be drawn, P-2-3-4, P-1-3-4, P-1-2-4, and P-1-2-3.

Let V be the volume of tetrahedron 1-2-3-4.
Let V_1 be the volume of tetrahedron P-2-3-4.
Let V_2 be the volume of tetrahedron P-1-3-4.
Let V_3 be the volume of tetrahedron P-1-2-4.
Let V_4 be the volume of tetrahedron P-1-2-3.

$$\xi_1 = V_1/V$$
$$\xi_2 = V_2/V$$
$$\xi_3 = V_3/V$$
$$\xi_4 = V_4/V$$

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_2 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}$$

$$6V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_2 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Shape Functions:

$$\xi_1 = N_1$$
$$\xi_2 = N_2$$
$$\xi_3 = N_3$$
$$\xi_4 = N_4$$

$$\phi = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3 + N_4 \phi_4$$

Integration

$$\iiint_V \xi_1^k \xi_2^l \xi_3^m \xi_4^n \, dV = 6V \frac{k!\, l!\, m!\, n!}{(3+k+l+m+n)!}$$

Centroidal Coordinates

$$\iiint_{x^r y^s} x^r y^s z^t dV = \frac{V}{20}\sum_{i=1}^{4} x_i^r y_i^s z_i^t$$

$$\left(\text{if } r+s+t=2 \text{ and centroid is at } x=y=z=0\right)$$

$$\xi_1 = s$$

$$\xi_2 = t$$

$$\xi_3 = z'$$

$$\xi_4 = 1 - s - t - z'$$

Chain Rule

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi_1}\frac{\partial\xi_1}{\partial x} + \frac{\partial\phi}{\partial\xi_2}\frac{\partial\xi_2}{\partial x} + \frac{\partial\phi}{\partial\xi_3}\frac{\partial\xi_3}{\partial x} + \frac{\partial\phi}{\partial\xi_4}\frac{\partial\xi_4}{\partial x}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial\xi_1}\frac{\partial\xi_1}{\partial y} + \frac{\partial\phi}{\partial\xi_2}\frac{\partial\xi_2}{\partial y} + \frac{\partial\phi}{\partial\xi_3}\frac{\partial\xi_3}{\partial y} + \frac{\partial\phi}{\partial\xi_4}\frac{\partial\xi_4}{\partial y}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial\xi_1}\frac{\partial\xi_1}{\partial z} + \frac{\partial\phi}{\partial\xi_2}\frac{\partial\xi_2}{\partial z} + \frac{\partial\phi}{\partial\xi_3}\frac{\partial\xi_3}{\partial z} + \frac{\partial\phi}{\partial\xi_4}\frac{\partial\xi_4}{\partial z}$$

Chain Rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z'}$$

$$\frac{\partial f}{\partial x}=\frac{\begin{vmatrix}\frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial f}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}}{\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}} \qquad \frac{\partial f}{\partial x}=\frac{\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial f}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}}{\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}} \qquad \frac{\partial f}{\partial x}=\frac{\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial f}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial f}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial f}{\partial z'}\end{vmatrix}}{\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}}$$

Jacobian matrix:

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{bmatrix}$$

Strains in terms of an operator matrix:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=\begin{bmatrix}\frac{\partial(\ )}{\partial x} & 0 & \\ 0 & \frac{\partial(\ )}{\partial y} & \\ & & \frac{\partial(\ )}{\partial z}\\ \frac{\partial(\ )}{\partial y} & \frac{\partial(\ )}{\partial x} & \\ & \frac{\partial(\ )}{\partial z} & \frac{\partial(\ )}{\partial y}\\ \frac{\partial(\ )}{\partial z} & & \frac{\partial(\ )}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}$$

$$\frac{\partial(\ )}{\partial x}=\frac{1}{|J|}\begin{vmatrix}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial(\ )}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial(\ )}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix} \qquad \frac{\partial(\ )}{\partial y}=\frac{1}{|J|}\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial(\ )}{\partial s} & \frac{\partial z}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial(\ )}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial(\ )}{\partial z'} & \frac{\partial z}{\partial z'}\end{vmatrix}$$

$$\frac{\partial(\ )}{\partial z}=\frac{1}{|J|}\begin{vmatrix}\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial(\ )}{\partial s}\\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial(\ )}{\partial t}\\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial(\ )}{\partial z'}\end{vmatrix}$$